THESIS

CONTRIBUTIONS
TO
BORN'S FIELD THEORY

BY

B.S. Madhava Rao.,

Contributions to Born's Field Theory.

THESIS

submitted for the degree of Doctor of Science
of the University of Calcutta

by

B. S. Madhava Rao, M.Sc., F.R.A.S.,
Professor of Mathematics,
University of Mysore.

C O N T E N T S.

		Pages.
	Preface.	i - vii
(1)	Ring - Singularity in Born's Unitary theory - I	1-22
(2)	" II (Parts A and B)	23-49
(3)	On the fine structure of Balmer lines	50-56
(4)	Semi-vectors in Born's field theory.	57-72
(5)	Theorem on action functions in Born's field theory	73-77
(6)	Generalised action-functions in Born's Electro-dynamics	78-109
(7)	Complex Representation in Born's Field Theory - I	110-124
(8)	" II (Biquaternion representation).	125-135

PREFACE.

The thesis presented consists of eight papers of which five and part of another have already been published, one is to be shortly published and the rest have been given here for the first time. All the papers are devoted, exclusively, to the classical aspect of Born's field theory.

It is perhaps not necessary to state in this preface the salient features of this new field theory. An excellent and short account is given by Heitler in his book "Quantum Theory of Radiation" and Born himself has given an elementary and illuminating account in his article "The Mysterious Number,137" (Proc. Ind.Acad. Sci., A.Vol.2 (1935), p.549 - 61). It may however be mentioned that while the classical part of the theory is very satisfactory, the quantisation does not appear to be free from fundamental difficulties. This is not very surprising either, for, recent work in positron theory and theory of cosmic showers has shown that it is quite important to introduce a characteristic length in the early stages of an Electro-dynamical theory i.e. in the classical part. This is exactly what Born's field theory does. The knotty questions relating to "infinite self-energy" and "infinite self-force" which

are consequences of the non-introduction of such a length, are easily eliminated in this theory. Also the old question of structure of the electron and the associated question of cohesive forces of non-electromagnetic origin attacked in vain by Abraham, Poincaré and others receive elegant solutions, thanks to the non-linear character of the theory. It can safely be said that, whatever its achievements might be hereafter, the classical aspect of Born's field theory has come to acquire a permanent value as a new approach to problems of electrodynamics dealing with very high field strengths. It appears therefore worth while to carry out investigations in this part of the theory and tackle problems amenable to it. Another reason, which at present, remains only a hope, for carrying out comprehensive work in this direction is the expectation that when the recently introduced Neutrino theory of light initiated by Jordan and, in another form, by De Broglie is developed so as to yield the field equations of electro-dynamics these latter will be in the form given by Born's theory.

The eight papers in this thesis could be roughly classified into two divisions, the first three being related to the physical aspect of the theory and the rest to the mathematical aspect. The first two papers add a

new type of problem that can be tackled by means of Born's field theory. The only problem so far discussed in the theory in relation to the condition of "finite self-energy" is the special case of an electrostatic field with central symmetry. These two papers add the case of axial symmetry in the zero-approximation, i.e. the assumption that the field theory solutions valid in the immediate neighbourhood of the axial structure are to be super-posed on the Maxwellian solutions to calculate the self-energy. The method has been worked out with the several types of Born's Electro-dynamics and gives rise to concordant results proving the soundness of the method. It also suggests that Kramer's idea of explaining the spin of the electron on purely classical notions is not consistent with the principles of the unitary field theory and points out that all models, except perhaps that of the point-singularity, are unsuitable for the representation of the elementary particles. The third paper which treats of the radiation perturbation in the central field of the one-electron problem serves to bring out clearly the characteristic feature of Born's theory which derives, purely classically, several results

which could be dealt with by quatum methods. It has been shown by Meixner (Ann.der Phys.1935,23,371 and 1936,27,389) that the effect of the interaction of the electron and its radiation field could be accounted for on Dirac's theory of radiation by using the methods of Weisskopf and Wigner. The work in this paper leads to identical conclusions by using the totally different classical method of Born's theory in a very much simpler way. Another case where this feature of Born's field theory is exhibited is the case of scattering of light by light, first suggested by Debye. Euler and Köckel have explained this on Diracs positron theory and also indicated the parallelism with Born's theory. I have shown in the sixth paper how this parallelism can be carried still further to a general type of Born's Electro-dynamics.

The remaining papers of a mathematical nature deal with questions of relativistic invariance, several types of representations of the theory and of generalised Electro-dynamics. The fourth paper supplies a necessary proof of relativistic (Lorentz) invariance of the field equations and for this purpose the ordinary tensor theory is found insufficient and use is made of Einstein-Mayer's semi-vector theory. This paper is

perhaps the unique case where semi-vectors (as opposed to spinors) are applied in any particular investigation of this nature.

The fifth paper proves the uniqueness of Born's action-function under specified conditions. The sixth paper on general types of action functions in Born's electro-dynamics carries the work of Infeld one step further and adds one more degree of freedom in the choice of suitable action-functions for the theory. While the physical significance of this ambiguity is not quite clear, the added degree of freedom greatly enhances the beauty of the mathematical formalism. This generalisation has brought out close connections with the complex formalism of the theory developed by Weiss and goes quite deep into the several invariants of Born's Electro-dynamics. In particular it might be noticed that while the original Born-Infeld action function was in an irrational (square-root) form, and Schrodinger's representation gave it a rational form, the present paper gives a form expressed in terms of transcendental functions.

The last two papers are concerned with the complex representations of the theory. The seventh paper is confined to Schrodinger's representation and

extends his results to the υ and ν - representations wherein the Lagrangian takes, instead of the rational form of Schrodinger, again the square root form of Born's earliest expression. The last paper on biquaternion representation extends the results of Watson by means of alternative representations, and the application of biquaternions to generalised action functions, introducing in particular the notion of logarithm of a biquaternion for a certain complex invariant.

The last five papers serve to bring out the richness and variety from a mathematical point of view of the methods of Born's Electro-dynamics, and may perhaps prove useful in the case of an eventual application of the considerations treated in them to quantum electrodynamics.

I have benefitted very greatly in the preparation of this thesis to the published works of Born, Infeld, Pryce, Hoffmann, Weiss, and Watson from which I have drawn freely. My acknowledgements to Prof.Born's help will be found in some of the papers composing this thesis. I am greatly indebted to him for his encouragement and kind interest in my work.

I have not thought it necessary to append a

bibliography at the end of the Thesis since copious references to relevant literature on the subject are given in the footnotes of the individual papers.

P.T.O

(1)

RING - SINGULARITY IN BORN'S UNITARY THEORY - I.

Reprinted from "The Proceedings of the Indian Academy of Sciences",
Vol. IV, No. 3, Sec. A, 1936.

RING-SINGULARITY IN BORN'S UNITARY THEORY—I.

BY B. S. MADHAVA RAO.
(University of Mysore.)
(From the Department of Physics, Indian Institute of Science, Bangalore.)

Received August 17, 1936.

1. *Introduction.*

IN two papers on the unitary theory of field and matter Born[1] has given a classical treatment of the derivation of the equations of motion of a point-singularity representing a particle. The chief assumption made is that the external field must be constant over the " diameter " of the particle. The method is to start from a variation principle representing both the motion of the field and the singularity, the latter part also being written as a space-time integral involving Dirac's δ-function. In addition Born has also introduced a spin for the particle and given a derivation of Kramers'[2] formulæ on the basis of the unitary theory. A treatment on similar lines without however introducing the spin has also been given by Pryce[3] who starts with the Hamiltonian instead of the Lagrangian.

In the second of the papers referred to above, Born has shown that the results obtained by Kramers are difficult to interpret on the basis of the unitary theory and shown that the difficulty disappears if the particle be assumed to have, besides the magnetic moment, a charge or an electric moment or both. This enables the derivation of a set of equations of motion which are a generalisation of Kramers' equations, self-consistent without imposing a restriction on the Lagrangian and satisfactory from the purely formal standpoint. Since, however, the elementary particles occurring in nature have no electric rest-moment the conclusion was drawn that, on the basis of the unitary theory, point-singularities were not the correct representation of the particles. It was finally suggested that there might be other possibilities.

I have considered in this paper an elementary particle as a ring-singularity. The principal results obtained can be summarised as under :—

(*a*) Finite expressions are obtained for the total energy and angular momentum.

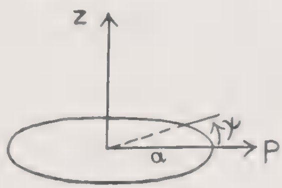
[1] M. Born, *Proc. Ind. Acad. Sci.*, A, 1936, 3, 8; *ibid.*, 1936, A, 3, 85.
[2] H. A. Kramers, *Physica*, 1934, 1, 825; Zeeman, *Verhandelingen*, 1935, p. 403.
[3] M. H. L. Pryce, *Proc. Roy. Soc.*, A, 1936, **155**, 597.

(*b*) The ratio of the magnetic moment to angular momentum is obtained as $e/2\mu$ and not e/μ (μ = rest-energy) showing that we do not obtain an explanation of the spin.

(*c*) Taking the ring-singularity as representing a proton we can derive an estimate of the ring-radius so as to explain the high mass of the proton.

2. *Field Equations.*

Let the axis of symmetry be taken as the z-axis and the radius of the ring a. We introduce cylindrical co-ordinates ρ, ψ, z and assume axial symmetry. The linear density is given by $\eta = e/2\pi a$ and $\vec{v}$ denotes the velocity vector. We introduce the Dirac δ-function such that $\delta = \infty$ on the ring and



$$\int \delta d\rho dz = 1.$$

Proceeding as in the papers of Born quoted above[4] without using spin considerations we obtain the field equations for the stationary case in the form.

$$(2,\ 1) \qquad \begin{cases} \text{rot } \vec{H} = \eta\vec{v}\delta\ ; & \text{div } \vec{D} = \eta\delta\ ; \\ \text{div } \vec{B} = 0\ ; & \text{rot } \vec{E} = 0. \end{cases}$$

We shall now obtain the equations of transformation for vector-components in the case of axial symmetry and also derive expressions for the operators *div*, and *rot*.[5] From the equations of transformation,

$$\left.\begin{aligned} x &= \rho \cos \psi \\ y &= \rho \sin \psi \\ z &= z \end{aligned}\right\} \tan \psi = \frac{y}{x},\ \frac{\partial \psi}{\partial x} = -\,y/\rho^2 \text{ and } \frac{\partial \psi}{\partial y} = \frac{x}{\rho^2}$$

one has

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{x}{\rho}\frac{\partial}{\partial \rho} - \frac{y}{\rho^2}\frac{\partial}{\partial \psi} \\ \frac{\partial}{\partial y} &= \frac{y}{\rho}\frac{\partial}{\partial \rho} + \frac{x}{\rho^2}\frac{\partial}{\partial \psi} \end{aligned}\right\} \text{ and } \frac{\partial}{\partial \rho}\frac{x}{\rho} = 0,\ \frac{\partial}{\partial \rho}\frac{y}{\rho} = 0$$

and therefore for any scalar ϕ of cylindrical symmetry $\left(\frac{\partial \phi}{\partial \psi} = 0\right)$

$$(2,\ 2) \qquad \frac{\partial \phi}{\partial x} = \frac{x}{\rho}\frac{\partial \phi}{\partial \rho}\ ;\ \frac{\partial \phi}{\partial y} = \frac{y}{\rho}\frac{\partial \phi}{\partial \rho}$$

---

[4] See Born, I, p. 10, equations (1, 9).

[5] I have to thank Dr. Beth, one of Prof. Born's pupils, for having communicated to me the correct proofs of the derivation of these expressions and of (2, 6).

and for any vector $\vec{A}$,

$$(2,3)\qquad \begin{cases} A_x = \dfrac{x}{\rho} A_\rho - \dfrac{y}{\rho} A_\psi \\[2ex] A_y = \dfrac{y}{\rho} A_\rho + \dfrac{x}{\rho} A_\psi \end{cases} \left| \begin{array}{l} A_\rho = \dfrac{x}{\rho} A_x + \dfrac{y}{\rho} A_y \\[2ex] A_\psi = -\dfrac{y}{\rho} A_x + \dfrac{x}{\rho} A_y \end{array} \right.$$

Accordingly

$$\left.\begin{array}{l} \mathrm{grad}_\rho\, \phi = \dfrac{\partial \phi}{\partial \rho} \\[2ex] \mathrm{grad}_\psi\, \phi = 0 \\[2ex] \mathrm{grad}_z\, \phi = \dfrac{\partial \phi}{\partial z} \end{array}\right\}$$

so that if a vector be a gradient it has no ψ-component.

To calculate div $\vec{A}$ for any arbitrary vector $\vec{A}$, we divide the vector field into two fields: $\vec{A} = \vec{B} + \vec{C}$, such that $B_\psi \equiv 0$, $C_\rho \equiv 0$, $C_z \equiv 0$, *i.e.*, $|C| = C = C_\psi$. We can prove that div $\vec{C} = 0$, for

$$\mathrm{div}\, \vec{C} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y}$$

$$\frac{\partial C_x}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{y}{\rho} C\right] = -\frac{x}{\rho}\frac{\partial}{\partial \rho}\left(\frac{y}{\rho} C\right) + \frac{y}{\rho^2}\frac{\partial}{\partial \psi}\left(\frac{y}{\rho} C\right)$$

$$= -\frac{xy}{\rho^2}\frac{\partial C}{\partial \rho} + \frac{y}{\rho^2} C \frac{\partial}{\partial \psi}\frac{y}{\rho}$$

$$= -\frac{xy}{\rho^2} + \frac{xy}{\rho^3} C.$$

$$\frac{\partial C_y}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x}{\rho} C\right] = \frac{y}{\rho}\frac{\partial}{\partial \rho}\left(\frac{x}{\rho} C\right) + \frac{x}{\rho^2}\frac{\partial}{\partial \psi}\left(\frac{x}{\rho} C\right)$$

$$= \frac{xy}{\rho^2}\frac{\partial C}{\partial \rho} - \frac{xy}{\rho^3} C$$

$$\therefore\ \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} = 0,\ \textit{i.e.},\ \mathrm{div}\, \vec{C} = 0 \text{ and } \mathrm{div}\, \vec{A} = \mathrm{div}\, \vec{B};$$

but, $\mathrm{div}\, \vec{B} = \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} + \dfrac{\partial B_z}{\partial z}$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\rho} B_\rho\right) + \frac{\partial}{\partial y}\left(\frac{y}{\rho} B_\rho\right) + \frac{\partial B_z}{\partial z}, \text{ using (2, 3) since } B_\psi = 0$$

$$= \frac{B_\rho}{\rho} + x\frac{\partial}{\partial x}\left(\frac{B_\rho}{\rho}\right) + \frac{B_\rho}{\rho} + y\frac{\partial}{\partial y}\left(\frac{B_\rho}{\rho}\right) + \frac{\partial B_z}{\partial z}$$

$$= \frac{2B_\rho}{\rho} + x\frac{\partial}{\partial \rho}\left(\frac{B_\rho}{\rho}\right)\frac{\partial \rho}{\partial x} + y\frac{\partial}{\partial \rho}\left(\frac{B_\rho}{\rho}\right)\frac{\partial \rho}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$= \frac{2B_\rho}{\rho} + \rho \frac{\partial}{\partial \rho}\left(\frac{B_\rho}{\rho}\right) + \frac{\partial B_z}{\partial z} \left[\text{since from } \rho^2 = x^2 + y^2 \right.$$

$$x \frac{\partial \rho}{\partial x} + y \frac{\partial \rho}{\partial y} = \rho$$

$$= \frac{B_\rho}{\rho} + \frac{\partial B_\rho}{\partial \rho} + \frac{\partial B_z}{\partial z}$$

Hence, for any arbitrary vector $\vec{A}$

$$(2,\ 4) \qquad \operatorname{div} \vec{A} = \frac{1}{\rho}\left\{\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{\partial(\rho A_z)}{\partial z}\right\}.$$

We shall now calculate rot $\vec{A}$ for an arbitrary vector $\vec{A}$ which is however axially-symmetric, *i.e.*, for which

$$\frac{\partial A_\rho}{\partial \psi} = \frac{\partial A_\psi}{\partial \psi} = \frac{\partial A_z}{\partial \psi} = 0.$$

$$\operatorname{rot}_x \vec{A} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{y}{\rho}\frac{\partial A_z}{\partial \rho} - \frac{y}{\rho}\frac{\partial A_\rho}{\partial z} - \frac{x}{\rho}\frac{\partial A_\psi}{\partial z}$$

$$\operatorname{rot}_y \vec{A} = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \frac{x}{\rho}\frac{\partial A_\rho}{\partial z} - \frac{y}{\rho}\frac{\partial A_\psi}{\partial z} - \frac{x}{\rho}\frac{\partial A_z}{\partial \rho}$$

$$\operatorname{rot}_z \vec{A} = \frac{x}{\rho}\frac{\partial}{\partial \rho}\left\{\frac{y}{\rho}A_\rho + \frac{x}{\rho}A_\psi\right\} - \frac{y}{\rho^2}\frac{\partial}{\partial \psi}\left\{\frac{y}{\rho}A_\rho + \frac{x}{\rho}A_\psi\right\}$$

$$- \frac{y}{\rho}\frac{\partial}{\partial \rho}\left\{\frac{x}{\rho}A_\rho - \frac{y}{\rho}A_\psi\right\} - \frac{x}{\rho^2}\frac{\partial}{\partial \psi}\left\{\frac{x}{\rho}A_\rho - \frac{y}{\rho}A_\psi\right\}$$

$$= \frac{\partial A_\psi}{\partial \rho} + \frac{1}{\rho}A_\psi.$$

Hence,

$$(2,\ 5) \qquad \begin{cases} \operatorname{rot}_\rho \vec{A} = -\dfrac{\partial A_\psi}{\partial z} \\ \operatorname{rot}_\psi \vec{A} = \dfrac{\partial A_\rho}{\partial z} - \dfrac{\partial A_z}{\partial \rho} \\ \operatorname{rot}_z \vec{A} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho A_\psi) \end{cases}$$

We shall now prove that for the field vectors $\vec{E}$, $\vec{H}$, $\vec{B}$, $\vec{D}$ satisfying the differential equations (2, 1), we have the relations

$$(2,\ 6) \qquad E_\psi = 0, \quad H_\psi = 0, \quad B_\psi = 0, \quad D_\psi = 0.$$

As regards $\vec{E}$, (2, 1) shows that it is irrotational and has no singularities. Hence it is the gradient of a non-singular scalar and from the theorem proved on p. 357, it follows that $E_\psi = 0$.

Next considering $\vec{H}$, we observe that it has a singularity and cannot be expressed as the gradient of an analytic scalar function. We write

$\vec{H} = \vec{H}_{cl} + \vec{H}^1$, the first term being the classical part on account of rot $\vec{H} = 0$ and rot $\vec{H}_{cl} = 0$, we get rot $\vec{H}^1 = 0$. Also $\vec{H}_{cl}$ has the same singularities as $\vec{H}$ and accordingly $\vec{H}^1$ has no singularities. This fact along with rot $\vec{H}^1 = 0$ enables us to deduce as above that $\vec{H}^1$ has no ψ-component. But $\vec{H}_{cl}$ is well-known and has no ψ-component. Accordingly $H_\psi = 0$.

The proof for $\vec{B}$ is obtained by considering

$$\vec{H} = \frac{\partial L}{\partial \vec{B}} \text{ where } L = L\,(F,\, G) \text{ is the Lagrangian}$$

$$i.e., \quad \vec{H} = L_F \frac{\partial F}{\partial \vec{B}} + L_G \frac{\partial G}{\partial \vec{B}} = L_F\, \vec{B} + L_G\, \vec{E}$$

Neither $\vec{H}$ nor $\vec{E}$ has a ψ-component. Hence at all places where $L_F \neq 0$, $\vec{B}$ also has no ψ-component.

A similar proof for $\vec{D}$ is obtained from

$$\vec{D} = -\frac{\partial L}{\partial \vec{E}} = -L_F \frac{\partial F}{\partial \vec{E}} - L_G \frac{\partial G}{\partial \vec{E}} = L_F\, \vec{E} - L_G\, \vec{B}$$

$\vec{E}$ has no ψ-component and $\vec{B}$ none when $L_F \neq 0$. Hence $\vec{D}$ has no ψ-component at all places where $L_F \neq 0$ or $L_G = 0$.

Using (2, 4), (2, 5) and (2, 6) we can now write the field equation (2, 1) in the form

$$(2,\,7) \qquad \begin{cases} \dfrac{\partial H_\rho}{\partial z} - \dfrac{\partial H_z}{\partial \rho} = \eta v\, \delta\,; \quad \dfrac{1}{\rho}\left\{\dfrac{\partial(\rho D_\rho)}{\partial \rho} + \dfrac{\partial(\rho D_z)}{\partial z}\right\} = \eta\delta \\[2ex] \dfrac{\partial(\rho B_\rho)}{\partial \rho} + \dfrac{\partial(\rho B_z)}{\partial_z} = 0, \quad \dfrac{\partial E_z}{\partial \rho} - \dfrac{\partial E_\rho}{\partial z} = 0 \end{cases}$$

since, further, in the stationary case $v_\rho = v_z = 0$, $v_\psi = v$. The Lagrangian is given by

$$(2,\,8) \quad L = b^2 \left\{ \sqrt{1 + \frac{1}{b^2}\,(B_\rho{}^2 + B_z{}^2 - E_\rho{}^2 - E_z{}^2) - \frac{1}{b^4}\,(B_\rho E_\rho + B_z E_z)^2} - 1 \right\}$$

3. *Boundary conditions at $\rho = a$.*

Integrating the field equations over the element of volume

$$\iint \frac{1}{\rho}\left\{\frac{\partial(\rho D_\rho)}{\partial \rho} + \frac{\partial(\rho D_z)}{\partial z}\right\} 2\pi\rho\, d\rho\, dz = \iint \eta\delta \cdot 2\pi\rho\, d\rho\, dz = 2\pi a \eta = e$$

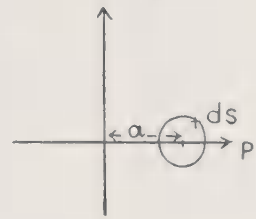
6

and the left-hand side

$$= 2\pi \iint \left\{\frac{\partial(\rho D_\rho)}{\partial\rho} + \frac{\partial(\rho D_z)}{\partial z}\right\} d\rho dz$$

$$= 2\pi \int_0 \rho \,\{D_\rho \cos(\rho, \nu) + D_z \cos(z, \nu)\}\, ds = 2\pi a \int_0 D_\nu ds$$

$$(3,\ 1)\qquad \left\{\begin{aligned} &i.e., \int_0 D_\nu ds = \frac{e}{2\pi a} = \eta \\ &\text{Similarly, } \int_0 B_\nu ds = 0 \end{aligned}\right.$$



Similarly

$$\iint \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial\rho}\right) 2\pi\rho d\rho dz = \iint \eta v \delta \cdot 2\pi\rho d\rho dz = 2\pi a v \eta = ev$$

and the left-hand side

$$= 2\pi a \int_0 \{H_\rho \cos(z, \nu) - H_z \cos(\rho, \nu)\}\, ds$$

$$= -2\pi a \int_0 H_s \, ds$$

$$(3,\ 2)\qquad \left\{\begin{aligned} &i.e., \int_0 H_s ds = -\eta v \\ &\text{Similarly, } \int_0 E_s ds = 0 \end{aligned}\right.$$

4. Solution in the case of zero-approximation.

Take $\rho = a$.

In this case the equations

$$(4,\ 1)\qquad \frac{\partial B_\rho}{\partial\rho} + \frac{\partial B_z}{\partial z} = 0\,;\ \frac{\partial E_z}{\partial\rho} - \frac{\partial E_\rho}{\partial z} = 0$$

are satisfied by taking $B_\rho = \lambda E_z$ and $B_z = -\lambda E_\rho$, so that

$$B_\rho E_\rho + B_z E_z = 0$$

and the Lagrangian

$$L = b^2 \left\{\sqrt{1 - \frac{1}{b^2}(1 - \lambda^2)(E_\rho^2 + E_z^2)} - 1\right\}$$

$$= b^2 \left\{\sqrt{1 + \frac{1}{b^2}\left(1 - \frac{1}{\lambda^2}\right)(B_\rho^2 + B_z^2)} - 1\right\}$$

or, without making these substitutions

$$L = b^2 \left\{\sqrt{1 + \frac{1}{b^2}(B_\rho^2 + B_z^2 - E_\rho^2 - E_z^2)} - 1\right\}$$

From this last form of L,

$$H_\rho = \frac{\partial L}{\partial B_\rho} = \frac{B_\rho}{\sqrt{1 + \frac{1}{b^2}(B_\rho^2 + B_z^2 - E_\rho^2 - E_z^2)}} = \frac{\lambda E_z}{\sqrt{\quad ,,}}$$

$$= -\lambda \frac{\partial L}{\partial E_z} = \lambda D_z$$

Similarly $H_z = -\lambda D_\rho$.

These relations could also have been deduced from the relations

$$\frac{\partial H_z}{\partial \rho} - \frac{\partial H_\rho}{\partial z} = 0\,; \; \frac{\partial D_\rho}{\partial \rho} + \frac{\partial D_z}{\partial z} = 0$$

and we would have obtained $H_\rho = \mu D_z$ and $H_z = -\mu D_\rho$; but, the above method however shows that $\mu = \lambda$.

We can determine λ from the boundary conditions

$$-\eta v = \int_0 H_s ds = \int_0 \{H_z \cos(\rho, \nu) - H_\rho \cos(z, \nu)\}\, ds$$

$$= -\lambda \int_0 \{D_\rho \cos(\rho, \nu) + D_z \cos(z, \nu)\}\, ds$$

$$= -\lambda \int D_\nu ds = -\lambda\eta$$

(4, 2) $\quad \therefore \quad \lambda = v.$ Let $\underline{1 - \lambda^2 = 1 - v^2 = a^2}.$

Rewriting the equations in the case of the zero-approximation

$$\frac{\partial D_\rho}{\partial \rho} + \frac{\partial D_z}{\partial z} = 0, \quad \frac{\partial E_z}{\partial \rho} - \frac{\partial E_\rho}{\partial z} = 0$$

with the boundary conditions

$$\int_0 D_\nu \, ds = \eta\,; \quad \int_0 E_s \, ds = 0$$

we find that a solution is given by

$$(4, 3) \qquad \left\{ \begin{aligned} D_\rho &= \frac{\eta}{2\pi r} \cdot \frac{\rho - a}{r} \\ D_z &= \frac{\eta}{2\pi r} \cdot \frac{z}{r} \end{aligned} \right\}$$

where $r^2 = (\rho - a)^2 + z^2$, *i.e.*, r is the distance from a point on the ring. We can easily express $\vec{E}$ in terms of $\vec{D}$ by considering the Hamiltonian H.

$$H = b^2 \left\{ \sqrt{1 + \frac{1}{b^2}(\vec{D}^2 - \vec{H}^2) - \frac{1}{b^4}(\vec{D}\,\vec{H})^2} - 1 \right\}$$

$$= b^2 \left\{ \sqrt{1 + \frac{1}{b^2}(\vec{D}^2 - \vec{H}^2)} - 1 \right\} \text{ since } (\vec{D}\,\vec{H}) = 0 \quad \text{in this case,}$$

as easily follows from the relation $H_\rho = \lambda D_z$ and $H_z = -\lambda D_\rho$.

$$\therefore\ E_\rho = \frac{\partial H}{\partial D_\rho} = \frac{D_\rho}{\sqrt{1 + \frac{1}{b^2}(\vec{D}^2 - \vec{H}^2)}} = \frac{D_\rho}{\sqrt{1 + \frac{a^2}{b^2}(D_\rho^2 + D_z^2)}}$$

and $$E_z = \frac{\partial H}{\partial D_z} = \frac{D_z}{\sqrt{1 + \frac{1}{b^2}(\vec{D}^2 - \vec{H}^2)}} = \frac{D_z}{\sqrt{1 + \frac{a^2}{b^2}(D_\rho^2 + D_z^2)}}$$

Putting in the values of D_ρ and D_z from (1)

$$(4,\ 4)\qquad \left\{\begin{aligned} E_\rho &= \frac{\eta}{2\pi r}\cdot\frac{\rho - a}{\sqrt{r^2 + r_1^2}} \\ E_z &= \frac{\eta}{2\pi r}\cdot\frac{z}{\sqrt{r^2 + r_1^2}} \end{aligned}\right\}$$

$$(4,\ 5)\qquad \text{where } r_1 = \frac{a\eta}{2\pi b}.$$

It is easy to verify that the E-equation, *viz.*, $\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = 0$ is satisfied by the values (2). This needs only showing that

$$(\rho - a)\frac{\partial}{\partial z}\left\{\frac{1}{r\sqrt{r^2 + r_1^2}}\right\} = z\frac{\partial}{\partial \rho}\left\{\frac{1}{r\sqrt{r^2 + r_1^2}}\right\}$$

$$i.e.,\ (\rho - a)\frac{\partial r}{\partial z} = z\frac{\partial r}{\partial \rho}$$

and this is true since $\frac{\partial r}{\partial z} = \frac{z}{r}$ and $\frac{\partial r}{\partial \rho} = \frac{\rho - a}{r}$.

From (2) $E = \frac{\eta}{2\pi}\cdot\frac{1}{\sqrt{r^2 + r_1^2}}$ and is finite for $r = 0$, having the value

$$(4,\ 6)\qquad \frac{\eta}{2\pi r_1} = \frac{b}{a} = b'.$$

We can now complete the solutions in the case of zero-approximation

$$(4,\ 7)\qquad \left\{\begin{aligned} H_\rho &= \frac{\eta v}{2\pi r}\cdot\frac{z}{r} \\ H_z &= -\frac{\eta v}{2\pi r}\cdot\frac{\rho - a}{r} \end{aligned}\right\}$$

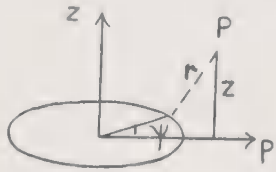
and

$$(4,\ 8)\qquad \left\{\begin{aligned} B_\rho &= \frac{\eta v}{2\pi r}\cdot\frac{Z}{\sqrt{r^2 + r_1^2}} \\ B_z &= -\frac{\eta v}{2\pi r}\cdot\frac{\rho - a}{\sqrt{r^2 + r_1^2}} \end{aligned}\right\}$$

9

5. *Solutions in the classical case.*

Treating the problem as the field due to a circular current, we have to determine the scalar and vector-potentials.



Referring to the adjoining figure the scalar potential ϕ is given by

$$\phi = \frac{1}{4\pi}\int_0^{2\pi} ad\psi \cdot \frac{\eta}{\sqrt{(\rho-a\cos\psi)^2 + a^2\sin^2\psi + z^2}}$$

$$i.e., \ \phi = \frac{a}{4\pi}\int_0^{2\pi} \frac{\eta d\psi}{\sqrt{\rho^2 + z^2 - 2\rho a\cos\psi + a^2}}$$

$$= \frac{\eta a}{4\pi}\int_{\pi/2}^{-\pi/2} \frac{-2d\psi_0}{R\sqrt{1-k^2\sin^2\psi_0}}\ ; \text{ where } R^2 = (\rho + a)^2 + z^2$$

$$k^2 = \frac{4\rho a}{R^2}\ ;\ \psi_0 = \frac{\pi - \psi}{2}$$

$$= \frac{\eta a}{\pi R}\int_0^{\pi/2} \frac{d\psi_0}{\sqrt{1 - k^2\sin^2\psi_0}}$$

$$(5,\ 1) \qquad \phi = a\eta\,\frac{K(k)}{\pi R} = \frac{e}{2\pi^2}\,\frac{K(k)}{R}$$

where $K(k)$ is the complete elliptic integral of the first kind.

Coming to a question of units, we observe that for $\rho,\ z \gg a$, (4) reduces to $\phi \rightarrow \dfrac{e}{4\pi R}$ since $K(0) = \pi/2$

i.e., the units we are using are the ordinary electrostatic units and *not* the Gaussian units. To transform to the latter we need only replace e by $4\pi e$. This fact shall be made use of later.

Coming to the vector-potential due to the circular current,[6] we have, denoting this by $\vec{A}$, $A_\rho = 0$, $A_z = 0$, and

$$(5,\ 2) \qquad A_\psi = \frac{\eta v}{\pi}\sqrt{\frac{a}{\rho}} \cdot \frac{1}{k}\left\{\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right\}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the 1st and 2nd kind.

---

[6] See Debye, *Ency. Math. Wiss.*, V., 2, § 17. p. 434.

From the potentials ϕ and $\vec{A}$, the field strengths are given by

$$\vec{E} = -\text{ grad }\phi\,; \quad \vec{H} = \text{rot }\vec{A}$$

$$\left.\begin{aligned} i.e.,\quad E_\rho &= -\frac{\partial\phi}{\partial\rho}\,;\; E_\psi = 0\,;\; E_z = -\frac{\partial\phi}{\partial z} \\ E_\rho &= -\frac{\partial A_\psi}{\partial z}\,;\; H_\psi = 0\,;\; H_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\psi) \end{aligned}\right\}$$

using (2, 5) and (2, 6)

From (4), we get

$$E_\rho = -\frac{\partial\phi}{\partial\rho} = \frac{a\eta}{\pi}\left[\frac{1}{R^2}\frac{\partial R}{\partial\rho}\cdot K(k) - \frac{1}{R}\cdot\frac{\partial K(k)}{\partial k}\cdot\frac{\partial k}{\partial\rho}\right]$$

$$R^2 = (\rho + a)^2 + Z^2\,;\; \frac{\partial R}{\partial\rho} = \frac{\rho + a}{R}\,;\; \frac{\partial R}{\partial z} = \frac{z}{R}$$

$$k^2 = \frac{4\rho a}{R^2}\,;\; \frac{1}{k}\frac{\partial R}{\partial\rho} = \frac{1}{2\rho} - \frac{\rho + a}{R^2}\,;\; \frac{1}{k}\frac{\partial k}{\partial z} = -\frac{z}{R^2}$$

From the theory of elliptic integrals,

$$\frac{\partial K(k)}{\partial k} = \frac{E(k)}{kk'^2} - \frac{K(k)}{k}$$

where k' is the modulus complementary to k, *i.e.*, $k'^2 = 1 - k^2$

$$i.e.,\;\; k'^2 = \frac{r^2}{R^2} \quad\text{or}\quad \underline{k' = \frac{r}{R}}$$

Hence,

$$E_\rho = \frac{a\eta}{\pi}\left[\frac{1}{R^2}\cdot\frac{\rho + a}{R}K(k) - \frac{1}{R}\left\{\left(\frac{E(k)R^2}{r^2} - K(k)\right)\left(\frac{1}{2\rho} - \frac{\rho + a}{R^2}\right)\right\}\right]$$

$$(5,\,3)\qquad i.e.,\;\left\{\begin{aligned} E_\rho &= \frac{a\eta}{2\pi R\rho}\left\{K(k) - \frac{Z^2 + a^2 - \rho^2}{r^2}E(k)\right\} \\ \text{Similarly}\quad E_z &= \frac{a\eta z}{\pi R r^2}E(k) \end{aligned}\right\}$$

Coming now to the magnetic field strength, we get from (5, 2)

$$H_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\psi) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\frac{\eta v}{2\pi}\cdot R\left\{\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right\}\right]$$

$$H_z = \frac{\eta v}{2\pi\rho}\cdot\frac{\partial}{\partial\rho}\left[R\left\{\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right\}\right] \qquad \left[\frac{\partial E(k)}{\partial k} = \frac{E(k) - K(k)}{k}\right.$$

$$= \frac{\eta v}{2\pi\rho}\left[\frac{\partial R}{\partial\rho}\left\{\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right\} + R\left\{\left(1 - \frac{k^2}{2}\right)\frac{\partial K(k)}{\partial k}\cdot\frac{\partial k}{\partial\rho} - K(k)\cdot k\frac{\partial k}{\partial\rho} - \frac{\partial E(k)}{\partial k}\cdot\frac{\partial k}{\partial\rho}\right\}\right]$$

$$= \frac{\eta v}{2\pi\rho}\left[\frac{\rho + a}{\mathrm{R}}\left\{\mathrm{K}(k) - \frac{\mathrm{E}(k)}{k'^2}\left(1 - \frac{k^2}{2}\right)\right\} + \frac{\mathrm{R}}{2\rho}\left\{\frac{\mathrm{E}(k)}{k'^2}\left(1 - \frac{k^2}{2} - k'^2\right) - \mathrm{K}(k)\frac{k^2}{2}\right\}\right]$$

Putting $k^2 = 4a\rho/\mathrm{R}^2$, $k'^2 = r^2/\mathrm{R}^2$, this reduces to

$$\mathrm{H}_z = \frac{\eta v}{2\pi\rho}\left[\frac{\rho}{\mathrm{R}}\,\mathrm{K}(k) + \mathrm{E}(k)\left\{\frac{a\mathrm{R}}{r^2} - \frac{(\rho + a)\,(\rho^2 + a^2 + z^2)}{\mathrm{R}r^2}\right\}\right]$$

$$= \frac{\eta v}{2\pi\rho}\left[\frac{\rho}{\mathrm{R}}\,\mathrm{K}(k) - \frac{\mathrm{E}(k)\rho}{\mathrm{R}r^2}\,(\rho^2 - a^2 + z^2)\right]$$

$$= \frac{\eta v\rho}{2\pi\rho\mathrm{R}}\left\{\mathrm{K}(k) - \frac{\rho^2 - a^2 + z^2}{r^2}\,\mathrm{E}(k)\right\}$$

Similarly,

$$\mathrm{H}_\rho = -\frac{\partial \mathrm{A}_\psi}{\partial z} = -\frac{\partial}{\partial z}\left[\frac{\eta v}{2\pi}\cdot\frac{\mathrm{R}}{\rho}\left\{\left(1 - \frac{k^2}{2}\right)\mathrm{K}(k) - \mathrm{E}(k)\right\}\right]$$

$$= -\frac{\eta v}{2\pi\rho}\frac{\partial}{\partial z}\left[\mathrm{R}\left\{\left(1 - \frac{k^2}{2}\right)\mathrm{K}(k) - \mathrm{E}(k)\right\}\right]$$

$$= -\frac{\eta v}{2\pi\rho}\left[\frac{\partial \mathrm{R}}{\partial z}\left\{\left(1 - \frac{k^2}{2}\right)\mathrm{K}(k) - \mathrm{E}(k)\right\} + \mathrm{R}\left\{\left(1 - \frac{k^2}{2}\right)\frac{\partial \mathrm{K}(k)}{\partial k}\cdot\frac{\partial k}{\partial z} - k\mathrm{K}(k)\frac{\partial k}{\partial z} - \frac{\partial \mathrm{E}(k)}{\partial k}\cdot\frac{\partial k}{\partial z}\right\}\right]$$

$$= -\frac{\eta v}{2\pi\rho}\left[\frac{\mathrm{Z}}{\mathrm{R}}\left\{\mathrm{K}(k) - \frac{\mathrm{E}(k)}{k'^2}\left(1 - \frac{k^2}{2}\right)\right\}\right]$$

simplifying by substituting the expressions for $\dfrac{\partial \mathrm{K}(k)}{\partial k}$ and $\dfrac{\partial \mathrm{E}(k)}{\partial k}$.

$$= -\frac{\eta v z}{2\pi\mathrm{R}\rho}\left\{\mathrm{K}(k) - \frac{\rho^2 + a^2 + \mathrm{Z}^2}{r^2}\,\mathrm{E}(k)\right\};$$

thus,

$$(5,\ 4)\qquad \begin{cases} \mathrm{H}_\rho = -\dfrac{\eta v z}{2\pi\mathrm{R}\rho}\left\{\mathrm{K}(k) - \dfrac{\rho^2 + a^2 + z^2}{r^2}\,\mathrm{E}(k)\right\} \\[2ex] \mathrm{H}_z = \dfrac{\eta v}{2\pi\mathrm{R}}\left\{\mathrm{K}(k) - \dfrac{\rho^2 + a^2 + z^2}{r^2}\right\}\mathrm{E}(k) \end{cases}$$

In (5, 3) and (5, 4) we can transform the expressions

$$\frac{z^2 + a^2 - \rho^2}{r^2},\quad \frac{\rho^2 + a^2 + z^2}{r^2},\quad \frac{\rho^2 - a^2 + z^2}{r^2}$$

by writing $z^2 = r^2 - (\rho - a)^2$ and obtain

$$(5,\ 5)\qquad \begin{cases} \mathrm{E}_\rho = \dfrac{a\eta}{\pi\mathrm{R}}\cdot\dfrac{\rho - a}{r^2}\,\mathrm{E}(k) + \dfrac{a\eta}{2\pi\mathrm{R}\rho}\,[\mathrm{K}(k) - \mathrm{E}(k)] \\[2ex] \mathrm{E}_z = \dfrac{a\eta}{\pi\mathrm{R}}\cdot\dfrac{z}{r^2}\,\mathrm{E}(k) \end{cases}$$

$$(5,\ 6)\qquad \begin{cases} H_\rho = \dfrac{a\eta v}{\pi R} \cdot \dfrac{z}{r^2}\,E(k) - \dfrac{\eta v z}{2\pi R\rho}\,[K(k) - E(k)] \\[2ex] H_z = -\dfrac{a\eta v}{\pi R} \cdot \dfrac{\rho - a}{r^2}\,E(k) + \dfrac{\eta v}{2\pi R}\,[K(k) - E(k)] \end{cases}$$

(5, 5) and (5, 6) are the exact classical solutions (where $\vec{B} = \vec{H}$ and $\vec{D} = \vec{E}$). Going to the limit $r \to 0$ which corresponds to the case $\rho = a$, $z = 0$, *i.e.*, on the ring, we have

$$R \to 2a,\ k' \to 0,\ \text{and}$$
$$E(k) \to 1 \ \text{and}\ K(k) \to \log((4/k').$$

Hence the classical solutions valid in the neighbourhood of the ring are

$$(5,\ 7)\qquad \begin{cases} E_\rho = \dfrac{\eta}{2\pi} \cdot \dfrac{\rho - a}{r^2} + \dfrac{\eta}{4\pi a}\left(\log \dfrac{8a}{r} - 1\right) \\[2ex] E_z = \dfrac{\eta}{2\pi} \cdot \dfrac{Z}{r^2} \end{cases}$$

and,

$$(5,\ 8)\qquad \begin{cases} H_\rho = \dfrac{\eta v}{2\pi} \cdot \dfrac{z}{r^2} \\[2ex] H_z = -\dfrac{\eta v}{2\pi} \cdot \dfrac{\rho - a}{r^2} + \dfrac{\eta v}{4\pi a}\left(\log \dfrac{8a}{r} - 1\right) \end{cases}$$

It is interesting to observe that the first terms in the right-hand members of (5, 7) and (5, 8) agree with the unitary field theory expression for the zero approximation case (4, 3), (4, 4), (4, 7), (4, 8) when, in the latter, we consider (r_1/r) small and neglect second and higher powers.

6. *Method of procedure for calculating integrals of energy and angular momentum.*

The remarks made above suggest the method of procedure to be adopted for the calculation of these integrals. We assume that the classical solutions hold good except in the immediate neighbourhood of the ring where we assume the field theory solutions (in §4) to be valid for values of r large compared with r_1. Thus, in the case of the total energy E, for example, we form the classical and field theory expressions U_{cl} and U_f and write

$$(6,\ 1)\qquad E = \int U_{cl}\, dV + U_f\, dV$$

Since we can certainly assume that most of the energy is concentrated in the neighbourhood of the ring we take the limits for r in the first integral r = a finite number = f (say) for the upper limit and r for the lower limit such that *r is small compared with a.* The limit f is further so chosen that no terms appear after integration which $\to \infty$ as $r \to \infty$. For the second

integral the lower limit is taken 0 while the upper limit r is assumed *large compared with* r_1 but small compared with a. The angular momentum is similarly treated. Further in calculating U_f we use the expressions in §4 and for U_{el} the expressions (5, 7) – (5, 8), or, in case a higher approximation is needed, the expressions (5, 5) and (5, 6) with the substitution therein, up to the desired degree of accuracy, from the series—expansions for K (k) and E (k)[7] valid in the neighbourhood of $k = 1$, or $k' = 0$, *viz.*,

$$(6,\ 2)\qquad \begin{cases} E(k) = 1 + \dfrac{k'^2}{2}\left(\log \dfrac{4}{k'} - \frac{1}{2}\right) + \dfrac{3}{16}k'^4\left(\log\dfrac{4}{k'} - 1 - \dfrac{1}{12}\right) + \cdots \\[2ex] K(k) = \log\dfrac{4}{k'} + \dfrac{k'^2}{2}\left(\log\dfrac{4}{k'} - 1\right) + \cdots \end{cases}$$

7. *Calculation of the energy.*

$$U_f = L + (\vec{D}\cdot\vec{E})$$

$$= b^2\left\{\sqrt{1 + \frac{1}{b^2}(\vec{B}^2 - \vec{E}^2)} - 1\right\} + \vec{D}\,\vec{E}, \text{ since } (\vec{B}\cdot\vec{E}) = 0 \text{ from §(4).}$$

$$= b^2\left\{\sqrt{1 + \frac{1}{b^2}\left(\frac{\eta^2 v^2}{4\pi^2 r^2}\cdot\frac{r^2}{r^2 + r_1^2} - \frac{\eta^2}{4\pi^2 r^2}\cdot\frac{r^2}{r^2 + r_1^2}\right)} - 1\right\} + \frac{\eta^2}{4\pi^2 r^2}\cdot\frac{1}{\sqrt{r^2 + r_1^2}}$$

$$= b^2\left\{\sqrt{1 - \frac{r_1^2}{r^2 + r_1^2}} - 1\right\} + \frac{b^2 r_1^2}{a^2 r\sqrt{r^2 + r_1^2}}, \text{ using (4, 5)}$$

$$\therefore \int U_f\,dV = 2\pi \iint U_f\,\rho\,d\rho\,dz$$

$$= 2\pi\int_0^r\int_0^{2\pi} U_f\,(a + r\cos\phi)\,r\,dr\,d\phi$$

$$= 2\pi a\int_0^r\int_0^{2\pi} U_f\,r\,dr\,d\phi$$

since $\int_0^{2\pi}\cos\phi\cdot d\phi = 0$, and U_f is a function of r only.

We have here taken r and ϕ to be polar co-ordinates in the meridian plane, such that,

$$(7,\ 1)\qquad \begin{cases} \rho - a = r\cos\phi \\ z = r\sin\phi \\ d\rho\,dz = r\,dr\,d\phi \end{cases}$$

---

[7] See, for example, Schlomilch, *Komp. d. Hoh. Anal.*, Bd. 2, pp. 322–323.

i.e., $\int U_f \, dV = 4\pi^2 a \int_0^r U_f \, r dr$

$$= 4\pi^2 ab^2 \int_0^r \left[\left\{ \sqrt{1 - \frac{r_1^2}{r^2 + r_1^2}} - 1 \right\} + \frac{r_1^2}{a^2 r \sqrt{r^2 + r_1^2}} \right] r dr$$

$$= 4\pi^2 ab^2 \int_0^r \left\{ \frac{r^2}{\sqrt{r^2 + r_1^2}} - r + \frac{r_1^2}{a^2 \sqrt{r^2 + r_1^2}} \right\} dr$$

$$= 4\pi^2 ab^2 \int_0^r \left(\sqrt{r^2 + r_1^2} - \frac{r_1^2}{\sqrt{r^2 + r_1^2}} - r \right) dr + 4\pi^2 ab'^2 \int_0^r \frac{r_1^2 \, dr}{\sqrt{r^2 + r_1^2}}$$

$$= 4\pi^2 ab^2 \int_0^r (\sqrt{r^2 + r_1^2} - r) \, dr + 4\pi^2 a \, (b'^2 - b^2) \int_0^r \frac{r_1^2 \, dr}{\sqrt{r^2 + r_1^2}}$$

$$= a\eta^2 \alpha^2 \int_0^x (\sqrt{1 + x^2} - x) \, dx + a\eta^2 v^2 \int_0^x \frac{dx}{\sqrt{1 + x^2}}$$

putting $r = r_1 x$ and using $r_1 = \dfrac{a\eta}{2\pi b}$; $b' = \dfrac{b}{a}$.

$$= \tfrac{1}{2} a\eta^2\alpha^2 \left\{ \frac{r}{r_1} \sqrt{1 + \frac{r^2}{r_1^2}} - \frac{r^2}{r_1^2} \right\} + \tfrac{1}{2} a\eta^2\beta^2 \log \left\{ \frac{r}{r_1} + \sqrt{1 + \frac{r^2}{r_1^2}} \right\}$$

(7. 2) where $\beta^2 = 1 + v^2$

(7, 3) $$\int U_f \, dV = a\eta^2 \log \left(\frac{2r}{r_1} \right)$$

where we have made use of the fact that r is large compared with r_1 and that *v is very nearly equal to the velocity of light, i.e., unity.*

We now proceed to calculate the classical part of the energy and write, making use of (5, 7)–(5, 8),

$$U_{cl} = \tfrac{1}{2} (\vec{E}^2 + \vec{H}^2) = \tfrac{1}{2} (\vec{E}_\rho^2 + E_z^2 + H_\rho^2 + H_z^2)$$

$$= \frac{\eta^2\beta^2}{8\pi^2 r^2} + \frac{\eta^2\beta^2}{32\pi^2 a^2} \left(\log \frac{8a}{r} \quad 1 \right)^2 + \frac{\eta^2\alpha^2}{8\pi^2 a} \cdot \frac{\rho - a}{r^2} \left(\log \frac{8a}{r} - 1 \right)$$

$$\therefore \int U_{cl} \, dV = \tfrac{1}{2} a\eta^2\beta^2 \int_r^f \frac{dr}{r} + \frac{\eta^2\beta^8}{8a} \int r \left(\log \frac{8a}{r} - 1 \right)^2 dr + \frac{\eta^2\alpha^2}{8\pi^2 a}$$

$$\iint \frac{\rho - a}{r^2} \left(\log \frac{8a}{r} - 1 \right) 2\pi\rho d\rho dz$$

$$\iint \frac{\rho - a}{r^2}\left(\log \frac{8a}{r} - 1\right) 2\pi\rho d\rho dz = \int_r^f \int_0^{2\pi} \frac{r \cos \phi}{r^2}\left(\log \frac{8a}{r} - 1\right)$$

$$2\pi\,(a + r \cos \phi)\; r dr d\phi$$

$$= \int_r^f \int_0^{2\pi} 2\pi r \left(\log \frac{8a}{r} - 1\right) \cos^2 \phi dr d\phi$$

$$= \int_r^f 2\pi^2 r \left(\log \frac{8a}{r} - 1\right) dr$$

Now,

$$\int r\left(\log \frac{8a}{r} - 1\right)^2 dr = \frac{r^2}{2}\left(\log \frac{8a}{r} - 1\right)^2 + \frac{r^2}{2}\left(\log \frac{8a}{r} - 1\right)$$

$$+ \frac{r^2}{4} = \frac{r^2}{2}\left(\log \frac{8a}{r}\right)^2 - \frac{r^2}{2} \log \frac{8a}{r} + \frac{r^2}{4}$$

and $\displaystyle\int r\left(\log \frac{8a}{r} - 1\right) dr = \frac{r^2}{2}\left(\log \frac{8a}{r} - 1\right) + \frac{r^2}{4} = \frac{r^2}{2}\left(\log \frac{8a}{r}\right) - \frac{r^2}{4}$

To avoid infinites at the upper limit we put $f = 8a$ and get

$$\int \mathrm{U}_{cl}\, d\mathrm{V} = \tfrac{1}{2}\, a\eta^2\beta^2 \log \frac{8a}{r} + 0\,(r \log r) + 0\,(r^2 \log r) + 0\,(r^2).$$

Since $\underset{r \to 0}{\mathrm{Lt}}\,(r \log r) = \underset{r \to 0}{\mathrm{Lt}}\,(r^2 \log r) = \underset{r \to 0}{\mathrm{Lt}}\,(r^2) = 0$ and r is small compared with a, we can neglect the last three terms and write

$$(7,\ 4) \qquad \int \mathrm{U}_{cl}\, d\mathrm{V} = \tfrac{1}{2}\, a\eta^2\beta^2 \log \frac{8a}{r} = a\eta^2 \log \frac{8a}{r}$$

From (7, 3) and (7, 4), we get

$$\mathrm{E} = a\eta^2 \log\left(\frac{2r}{r_1}\right) + a\eta^2 \log\left(\frac{8a}{r}\right)$$

$$(7,\ 5) \qquad i.e.,\ \mathrm{E} = a\eta^2 \log\left(\frac{16a}{r_1}\right)$$

giving a finite value for the energy.

8. *Calculation of the angular momentum.*

Corresponding to (6, 1) we write

$$(8,\ 1) \qquad \vec{\mathrm{M}} = \int \vec{\mathrm{M}}_{cl} d\mathrm{V} + \int \vec{\mathrm{M}}_f\, d\mathrm{V}.$$

$$\vec{\mathrm{M}} = \int \{\vec{r} \times (\vec{\mathrm{D}} \times \vec{\mathrm{B}})\}\, d\mathrm{V}$$

$$= \int \{\vec{\mathrm{D}}\,(\vec{r}\,\vec{\mathrm{B}}) - \vec{\mathrm{B}}\,(\vec{r}\,\vec{\mathrm{D}})\}\, d\mathrm{V}$$

$$\left.\begin{aligned} \therefore \quad M_\rho &= \int \{Z\,(D_\rho B_z - B_\rho D_z)\}\, dV \\ M_z &= \int \{\rho\,(D_z B_\rho - B_z D_\rho)\}\, dV \\ M_\psi &= 0 \end{aligned}\right\}$$

From the fact of axial symmetry, or observing that the integrand in M_ρ changes sign when z is replaced by $-z$, we have $M_\rho = 0$ and $M = M_z$. Hence

$$\int M_f\, dV = \int \rho\,(D_z B_\rho - B_z D_\rho)\, dV$$
$$= \frac{\eta^2 v}{4\pi^2} \int \frac{\rho}{r\sqrt{r^2 + r_1^2}}\, dV$$

using (4, 3) and (4, 8)

$$i.e., \quad \int M_f\, dV = \frac{\eta^2 v}{4\pi^2} \int\int \frac{\rho \cdot 2\pi\rho\, d\rho\, dz}{r\sqrt{r^2 + r_1^2}}$$

$$= \frac{\eta^2 v}{2\pi} \int_0^r \int_0^{2\pi} \frac{(a + r\cos\phi)^2\, r\, dr\, d\phi}{r\sqrt{r^2 + r_1^2}}$$

$$= \frac{\eta^2 v}{2\pi} \int_0^r \frac{dr}{\sqrt{r^2 + r_1^2}}\, (2\pi a^2 + \pi r^2)$$

$$= \eta^2 v \int_0^r \frac{a^2 dr}{\sqrt{r^2 + r_1^2}} + \frac{\eta^2 v}{2} \int_0^r \frac{r^2 dr}{\sqrt{r^2 + r_1^2}}$$

$$= \eta^2 v a^2 \int_0^r \frac{dr}{\sqrt{r^2 + r_1^2}} + \frac{\eta^2 v}{2} \int_0^r \sqrt{r^2 + r_1^2}\, dr - \frac{\eta^2 v r_1^2}{2} \int_0^r \frac{dr}{\sqrt{r^2 + r_1^2}}$$

$$= \frac{\eta^2 v}{2} \int_0^r \sqrt{r^2 + r_1^2}\, dr + \frac{\eta^2 v}{2}\,(2a^2 - r_1^2) \int_0^r \frac{dr}{\sqrt{r^2 + r_1^2}}$$

$$= \frac{\eta^2 v r_1^2}{2} \int_0^x \sqrt{1 + x^2}\, dx + \frac{\eta^2 v}{2}\,(2a^2 - r_1^2) \int_0^x \frac{dx}{\sqrt{1 + x^2}}$$

$$= \frac{\eta^2 v r_1^2}{2}\left[\tfrac{1}{2} x\sqrt{1 + x^2} + \tfrac{1}{2}\log\left(x + \sqrt{1 + x^2}\right)\right]$$
$$+ \frac{\eta^2 v}{2}\,(2a^2 - r_1^2)\log\left(x + \sqrt{1 + x^2}\right)$$

$$= \frac{\eta^2 v}{2}\left(2a^2 - \frac{r_1^2}{2}\right)\log\left(\frac{r}{r_1} + \sqrt{1 + \frac{r^2}{r_1^2}}\right) + \frac{\eta^2 v r\sqrt{r^2 + r_1^2}}{4}$$

$$= \eta^2 v a^2 \log\left(\frac{r}{r_1} + \sqrt{1 + \frac{r^2}{r_1^2}}\right) + \frac{\eta^2 v}{4}\, r\sqrt{r^2 + r_1^2}$$

neglecting the term in r_1^2

$$(8,\ 2) \qquad \int \mathrm{M}_f\, d\mathrm{V} = \eta^2 v a^2 \log\left(\frac{2r}{r_1}\right) + \frac{\eta^2 v}{4}\, r^2$$

since r is assumed large compared with r_1.

On the classical theory.

$$\mathrm{M}_z = \int \rho\, (\mathrm{E}_z\, \mathrm{H}_\rho - \mathrm{H}_z\, \mathrm{E}_\rho)\, d\mathrm{V}$$

Using the exact expressions (5, 5)–(5, 6) we have

$$\mathrm{E}_z\mathrm{H}_\rho - \mathrm{H}_z\mathrm{E}_\rho = \frac{a\eta}{\pi\mathrm{R}} \cdot \frac{z}{r^2}\,\mathrm{E}(k) \left\{\frac{a\eta v}{\pi\mathrm{R}} \cdot \frac{z}{r^2}\,\mathrm{E}(k) - \frac{\eta v z}{2\pi\mathrm{R}\rho}\,[\mathrm{K}(k) - \mathrm{E}(k)]\right\}$$

$$- \left\{\frac{a\eta}{\pi\mathrm{R}} \cdot \frac{\rho - a}{r^2}\,\mathrm{E}(k) + \frac{a\eta}{2\pi\mathrm{R}\rho}\Big(\mathrm{K}(k) - \mathrm{E}(k)\Big)\right\} \left\{-\frac{a\eta v}{\pi\mathrm{R}} \cdot \frac{\rho - a}{r^2}\,\mathrm{E}(k) + \frac{\eta v}{2\pi\mathrm{R}}\,\overline{\mathrm{K}(k) - \mathrm{E}(k)}\right\}$$

$$= \frac{a\eta^2 v}{\pi^2\mathrm{R}^2}\left[\left\{a\mathrm{E}^2\frac{z^2}{r^4} - \frac{1}{2\rho} \cdot \frac{z^2}{r^2}\,\mathrm{E}(\mathrm{K} - \mathrm{E})\right\} + \left\{\left(\frac{\rho - a}{r^2}\,\mathrm{E} + \frac{1}{2\rho}\,\overline{\mathrm{K} - \mathrm{E}}\right)\left(a\,\frac{\overline{\rho - a}}{r^2}\,\mathrm{E} - \tfrac{1}{2}\,\overline{\mathrm{K} - \mathrm{E}}\right)\right\}\right]$$

$$= \frac{a\eta^2 v}{\pi^2\mathrm{R}^2}\left[\left\{a\mathrm{E}^2\frac{z^2}{r^4} - \frac{1}{2\rho}\frac{z^2}{r^2}\,\mathrm{E}\,(\mathrm{K} - \mathrm{E})\right\} + \left\{a\mathrm{E}^2\frac{(\rho - a)^2}{r^4} + \mathrm{E}\,(\mathrm{K} - \mathrm{E})\left(\frac{a}{2\rho} \cdot \frac{\rho - a}{r^3} - \frac{\rho - a}{2r^2}\right) - \frac{1}{4\rho}\,(\mathrm{K} - \mathrm{E})^2\right\}\right]$$

$$= \frac{a\eta^2 v}{\pi^2\mathrm{R}^2}\left\{\frac{a\mathrm{E}^2}{r^2} - \frac{\mathrm{E}(\mathrm{K} - \mathrm{E})}{2\rho} - \frac{1}{4\rho}\,(\mathrm{K} - \mathrm{E})^2\right\}$$

$$= \frac{a\eta^2 v}{\pi^2\mathrm{R}^2}\left\{\frac{a\mathrm{E}^2}{r^2} + \frac{\mathrm{E}^2}{2\rho} - \frac{\mathrm{EK}}{2\rho} + \frac{\mathrm{EK}}{2\rho} - \frac{\mathrm{K}^2}{4\rho} - \frac{\mathrm{E}^2}{4\rho}\right\}$$

$$= \frac{a\eta^2 v}{\pi^2\mathrm{R}^2}\left\{\left(\frac{a}{r^2} + \frac{1}{4\rho}\right)\mathrm{E}^2 - \frac{1}{4\rho}\,\mathrm{K}^2\right\}$$

$$= \frac{a\eta^2 v}{4\pi^2\rho\mathrm{R}^2}\left\{\frac{\mathrm{R}^2}{r^2}\,\mathrm{E}^2 - \mathrm{K}^2\right\}$$

$$(8,\ 3) \quad \therefore \quad \rho\,(\mathrm{E}_z\mathrm{H}_\rho - \mathrm{H}_z\,\mathrm{E}_\rho) = \frac{a\eta^2 v}{4\pi^2\mathrm{R}^2}\left\{\frac{[\mathrm{E}(k)]^2}{k'^2} - [\mathrm{K}(k)]^2\right\}$$

Before proceeding further with the approximations for E (k) and K (k) valid in the neighbourhood of the ring, we might notice the very elegant form (8, 3) for the angular-momentum, this being the exact expression. It might also be observed that (8, 3) shows directly that $\mathrm{M}\rho$ is an odd function of z, for, the expression

$$\mathrm{E}_\rho\mathrm{H}_z - \mathrm{H}_\rho\mathrm{E}_z = \frac{a\eta^2 v}{4\pi^2\mathrm{R}^2}\left\{[\mathrm{K}(k)]^2 - \frac{[\mathrm{E}(k)]^2}{k'^2}\right\}$$

is an even function of z, since only squares of E (k), K (k), k' and R appear in it and further k, k', r and R are all even functions of Z. Hence the expression

$$Z\,(E_\rho H_z - H_\rho E_z) \text{ is an odd function of Z.}$$

We now proceed with the approximation of (8, 3) by writing

$$E\,(k) = 1 + \frac{k'^2}{2}\left(\log\frac{4}{k'} - \tfrac{1}{2}\right) + \frac{3}{16}k'^4\left(\log\frac{4}{k'} - 1 - \frac{1}{12}\right)$$

and $$K\,(k) = \log\frac{4}{k'} + \frac{k'^2}{4}\left(\log\frac{4}{k'} - 1\right),$$ and get

$$\frac{E^2}{k'^2} - K^2 = \frac{1}{k'^2} + \left\{\log\frac{4}{k'} - \tfrac{1}{2} - \left(\log\frac{4}{k'}\right)^2\right\} + k'^2\left\{-\tfrac{1}{4}\left(\log\frac{4}{k'}\right)^2 + \frac{5}{8}\log\frac{4}{k'} - \frac{11}{32}\right\}$$

and this does not depend on ϕ; hence

$$\int M_{cl}\, dV = \int \rho\,(E_z H_\rho - H_z E_\rho)\, dV$$

$$= a^2\eta^2 v \int_r^f \frac{1}{R^2}\left(\frac{R^2}{r^2}\,[E(k)]^2 - [K(k)]^2\right) rdr$$

$$= a^2\eta^2 v \int \left\{\frac{[E(k)]^2}{r^2} - \frac{[K(k)]^2}{R^2}\right\} rdr$$

Substituting the series-expansions for E (k), K (k) and R $= 2a$ and proceeding with the integration as in the case of the energy we take $f = 8a$. We neglect as before, the terms $(r \log r)$ and $(r^2 \log r)$ at the lower limit but retain the term in r^2, in view of the form of expression (8, 2). This gives

$$(8,\ 4) \qquad \int M_{cl}\, dV = a^2\eta^2 v \log\left(\frac{8a}{r}\right) - \frac{\eta^2 v}{4}\, r^2$$

From (8, 2) and (8, 4) we get easily

$$(8,\ 5) \qquad M = a^2\eta^2 v \log\,(16a/r_1)$$

giving *a finite value for the angular-momentum as well.*

[A simpler way of deducing (8, 5) would have been to neglect the terms in r^2 both in (8, 2) and (8, 4) since in both cases r^2 appears as an absolute term.]

9. *Magnetic moment.*

The magnetic moment can be calculated from the expression (5, 2) for the classical vector-potential, *viz.*,

$$A_\psi = \frac{\eta v}{\pi}\sqrt{\frac{a}{\rho}}\cdot\frac{1}{k}\left\{\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right\}$$

For values of $k \ll 1$, *i.e.*, at great distance from the ring,[8]

$$\begin{aligned} A_\psi &= \frac{\eta v}{32}\sqrt{\frac{a}{\rho}} \cdot R^3 \left\{1 + \frac{3}{4}k^2 + - - -\right\} \\ &= \frac{\eta v}{32}\sqrt{\frac{a}{\rho}} \cdot \frac{8\sqrt{a\rho}\cdot a\rho}{R^3} \left\{1 + \frac{3}{4}\cdot\frac{4\rho a}{R^2} + - - -\right\} \\ &= \frac{a^2\eta v}{4}\cdot\frac{\rho}{R^3}\left\{1 + \frac{3\rho a}{R^2} + - - -\right\} \end{aligned}$$

Hence the magnetic moment

$$(9,\ 1) \qquad m = \frac{a^2\eta v}{4} = \frac{a^2 e v}{8\pi a} = \frac{aev}{8\pi}$$

10. Spin.

We shall write the expressions for the energy, angular momentum and magnetic moment in Gaussian units since we have so far used only electrostatic units (also $c = 1$). This requires as remarked on p. 363, the replacing of e by $4\pi e$ *i.e.*, η by $4\pi\eta$.

$$(10,\ 1) \qquad E = a\eta^2 \log\left(\frac{16a}{r_1}\right) = \frac{ae^2}{4\pi^2 a^2}\log\left(\frac{16a}{r_1}\right) \longrightarrow \frac{4e^2}{a}\log\left(\frac{16a}{r_1}\right)$$

$$(10,\ 2) \qquad M = a^2\eta^2 v \log\left(\frac{16a}{r_1}\right) = \frac{e^2 v}{4\pi^2}\log\left(\frac{16a}{r_1}\right) \longrightarrow 4e^2 v \log\left(\frac{16a}{r_1}\right)$$

$$(10,\ 3) \qquad m = \frac{a^2\eta v}{4} = \frac{a^2 e v}{8\pi a} = \frac{aev}{8\pi} \longrightarrow \frac{aev}{2}$$

Hence $$\frac{E}{M} = \frac{1}{av} = \frac{e}{2m} \text{ and,}$$

putting $E = \mu c^2$ we have $E = \mu$ since $c = 1$, which gives

$$(10,\ 4) \qquad \frac{m}{M} = \frac{e}{2\mu}$$

showing that if the particle represented by the ring-singularity be an electron, *the spin of the electron is not explained* by our classical method of investigation.

11. The mass of the proton.

If we consider the ring-singularity to represent the proton it can be shown that we can derive an estimate of the ring-radius which would give the right proton mass.

Equating (10, 2) to $\hbar = h/2\pi$, we get

$$\frac{h}{2\pi} = 4e^2 v \log\left(\frac{16a}{r_1}\right)$$

---

[8] *Cf.* Debye, *ibid.*, p. 435.

where r_1 is given from (4, 5), using Gaussian-units, as

$$r_1 = \frac{ae}{\pi ab} = \sqrt{1-v^2} \cdot \frac{e}{\pi ba}$$

Introducing now the radius r_0 of the point charge of the field theory,[9] which is given by $e = br_0^2$, we have

$$r_1 = \sqrt{1-v^2}\,\frac{r_0^2}{\pi a}$$

and

$$\frac{16a}{r_1} = \frac{16\pi a^2}{r_0^2} \cdot \frac{1}{\sqrt{1-v^2}} = 16\pi \left(\frac{a}{r_0}\right)^2 \frac{1}{\sqrt{1-v^2}}$$

and (10, 2) gives

$$\frac{h}{2\pi} = 4e^2 v \log \left\{ 16\pi \left(\frac{a}{r_0}\right)^2 \cdot \frac{1}{\sqrt{1-v^2}} \right\}$$

Introducing c, instead of taking $c = 1$, this can be written in the form

$$(11,\ 1) \qquad \frac{\hbar c}{e^2} = \frac{4v}{c} \log \left\{ 16\pi \left(\frac{a}{r_0}\right)^2 \frac{1}{\sqrt{1-v^2/c^2}} \right\}$$

We can similarly write the expression for the energy (10, 1) in the form

$$(11,\ 2) \qquad \mathrm{E} = \frac{4e^2}{a} \log \left\{ 16\pi \left(\frac{a}{r_0}\right)^2 \frac{1}{\sqrt{1-v^2/c^2}} \right\}$$

Let $m_0 =$ mass of a point charge and $\mathrm{M}_0 =$ mass of the ring, which we now assume to represent the mass of the proton. We have[10]

$$(11,\ 3) \qquad 1 \cdot 236 \frac{e^2}{r_0} = m_0 c^2$$

Putting $\mathrm{E} = \mathrm{M}_0 c^2$ in (11, 2) and making use of (11, 3) we get

$$(11,\ 4) \qquad \frac{\mathrm{M}_0}{m_0} = \frac{\mathrm{E}}{1 \cdot 236 e^2/r_0} = \frac{4r_0}{1 \cdot 236 a} \log \left[\frac{16\pi}{a} \left(\frac{a}{r_0}\right)^2 \right]$$

From the equations (11, 1) and (11, 4) we can determine v/c and r_0/a if we observe that the left-hand side of (11, 1) is the fine structure constant whose value we shall take as 137 and that the left-hand side of (11, 4) is the ratio of the masses of proton and electron which we shall take as 1840. Thus

$$\frac{4r_0}{1 \cdot 236 a} \log \left[\frac{16\pi}{a} \left(\frac{a}{r_0}\right)^2 \right] = 1840 \text{ from (11, 4), or}$$

$$(11,\ 5) \qquad \frac{4r_0}{a} \log \left[\frac{16\pi}{a} \left(\frac{a}{r_0}\right)^2 \right] = 2274, \text{ and}$$

$$(11,\ 6) \qquad \frac{4v}{c} \log \left[\frac{16\pi}{a} \left(\frac{a}{r_0}\right)^2 \right] = 137$$

---

[9] Born and Infeld, *Proc. Roy. Soc.*, A, 1934, **144**, p. 439.

[10] *Ibid.*, p. 446, equation (8, 7).

21

We can solve the equations (11, 5) and (11, 6) for v/c and r_0/a. Denoting these by x and y, the above equations can be written in the form

$$\left.\begin{aligned} &(11,\,7)\quad 4x \log\left(\frac{16\pi}{x^2\sqrt{1-y^2}}\right) = 2274 \\ &(11,\,8)\quad 4y \log\left(\frac{16\pi}{x^2\sqrt{1-y^2}}\right) = 137 \end{aligned}\right\}$$

The second equation can be written in the form (using only the nearest integers)

$$\frac{16\pi}{x^2\sqrt{1-y^2}} = \epsilon^{34/y} \quad (\epsilon \text{ is the exponential})$$

$$(11,\,9)\quad x^2 = \frac{16\pi}{\sqrt{1-y^2}}\,\epsilon^{-34/y}$$

Dividing (11, 8) by (11, 7), we have

$$(11,\,10)\quad \frac{y}{x} = \frac{137}{2274} \text{ or } y = \frac{x}{17}$$

Hence eliminating x between (11, 9) and (11, 10)

$$y = \frac{4\sqrt{\pi}}{17}\cdot\frac{1}{(1-y^2)}\,{}^{\frac{1}{4}}\,\epsilon^{-17/y}$$

$$(11,\,11)\quad i.e.,\ y\,(1-y^2)^{\frac{1}{4}} = \frac{8}{11}\,\epsilon^{-17/y}$$

From this equation we can estimate how near the value of y is to unity. Putting $1 - y^2 = \epsilon_0{}^4$ (ϵ_0 an infinitesimal) the order of magnitude of ϵ_0 is determined by

$$\epsilon_0\,(1-\epsilon_0{}^4)^{\frac{1}{2}} = \frac{8}{11}\,\epsilon^{-17} \quad \epsilon_0{}^4 \sim \epsilon^{-68} \text{ or } (1 - v^2/c^2) \sim {}^{-68} \sim 10^{-29}$$

$$(11,\,12)\quad i.e.,\ \epsilon_0 \sim \epsilon^{-17} \text{ or } v/c \sim 1$$

showing that $y = v/c$ is *very nearly equal to unity*. From equation (11, 10) we can therefore write

$$(11,\,13)\quad x = r_0/a \sim 17.$$

This gives an *estimate of the ring-radius*[11] *adequate to explain the mass of the proton.*

[11] Dr. Beth has kindly written to me about an attempt to determine the value of a for which the energy has a minimum. For this purpose, he takes account of some additional terms which I have omitted in (7, 4) and in place of (7, 5) he takes:

$$E = \frac{4e^2}{a}\cdot\frac{1+v^2}{2}\log\left[\frac{16\pi}{a}\left(\frac{a}{r_0}\right)^2\right].$$

The minimising of E gives rise to a quintic equation which he has solved by a method of approximation, giving

$$a = r_0\cdot 10^{16\,\pm\,1.6}$$

a result which, as Prof. Born has pointed out to me in a letter, is of an impossible order of magnitude.

12. Conclusion.

I hope to return in a future paper to the treatment of the ring-singularity in the general case and deal with questions connected with the conservation laws and self force and the equations of motion in a constant external field.

It gives me the greatest pleasure to thank Prof. Max Born for suggesting the problem, constantly guiding and advising me while here and corresponding with me from Cambridge.

(2)

RING - SINGULARITY IN BORN'S UNITARY THEORY - II

--- 0 ---

(Part A of this paper has been published in the Proc.Ind.Acad.Sci.A.1937, 6, 129).

PART A

Calculations with different action functions.

1. Introduction.

In an earlier paper, I have considered a ring-singularity on Born's unitary theory as representing an elementary particle - an electron or a proton. In either case it is possible to obtain finite values for the energy and angular momentum and this is certainly due to the non-linear character of Born's electro-dynamics which solves at one stroke the knotty problem of cohesive forces on non-electromagnetic origin inherent in the old classical theories of structure and inertial mass of the electron. But further results were of a purely negative character. If the model were taken to represent an electron, the ratio of magnetic moment to angular momentum was obtained as $e/2\mu$ and not e/μ showing that an explanation of the spin was not forthcoming. If the model were taken to represent a proton it was shown possible to derive an estimate of the ring-radius adequate to explain the high mass of the proton, but the ring was not in equilibrium in this position.[2]

---

(1) B.S. Madhava Rao, Pro. Ind. Acad. Sci., (A), 1936, 4, 355. This paper will be referred to as I.
(2) See I; foot-note on p.375.

Throughout this above investigation, I used a particular form of Born's electro-dynamics, viz., the earliest form of Born's field theory.[3] Infeld has shown later[4] that it is possible to find an infinite number of action functions for the new field theory each of which gives simple algebraic relations between the $f_{\kappa l}$ and $p_{\kappa l}$-fields and each leading to a finite energy for an electric particle. Still more recently[5] Hoffmann and Infeld have imposed a regularity condition restricting the choice of the action function and obtained special functions suitable for the cases of special and general relativity.

I have attempted in this paper to carry through my previous investigation on the ring-singularity using, first, the Hoffmann-Infeld action function, and next the one-parameter group of action functions given by Infeld. In these cases the authors restrict themselves to the case where the invariant $G = (\vec{B}\vec{E})$ does not appear. It may, therefore, appear that this

---

(3) Born and Infeld; Pro.Roy.Soc., A,1934,144,p.439.

(4) Infeld; Proc. Camb. Phil.Soc.,1936,32,127;33,70.

(5) Hoffmann and Infeld; Phys.Rev.1937,51, p.765 - referred to as II.

action-function is suitable only for the electrostatic case where $\vec{B} = \vec{H} = 0$ But in my treatment of the ring-singularity I have used the field theory solutions only in the zero-approximation[6], and in this case the conditions $(\vec{B}\vec{E}) = 0$ and $(\vec{D}\vec{H}) = 0$ are satisfied though $\vec{B}$ and $\vec{H}$ do not vanish.

2. Field equations and boundary conditions

The fieldequations for axial symmetry and stationery state can be taken over completely from I in the form

$$\left.\begin{aligned} \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = \eta v \delta \quad ; \quad & \frac{1}{\rho}\left\{ \frac{\partial(\rho D_\rho)}{\partial \rho} + \frac{\partial(\rho D_z)}{\partial z} \right\} = \eta\delta \\ \frac{\partial(\rho B_\rho)}{\partial \rho} + \frac{\partial(\rho B_z)}{\partial z} = 0 \quad ; \quad & \frac{\partial E_z}{\partial \rho} - \frac{\partial E_\rho}{\partial z} = 0 \end{aligned}\right\} \qquad (2,1)$$

The Lagrangian is, however, given by

$$L = H + R$$

and the Hamiltonian, for the Hoffmann-Infeld case which we will consider first, by

$$H = \frac{1}{2}\log(1+P) \qquad (2,2)$$

the invariants $F = (\vec{B}^2 - \vec{E}^2)$, $P = (\vec{D}^2 - \vec{H}^2)$, $R = (\vec{B}\vec{H} - \vec{D}\vec{E})$

---

(6) See I - § 4, p.360

(7) See II - footnote on p.768.

being given by the relations

$$R^2 = -FP, \quad \text{and}, \ (-P/F)^{1/2} = 1+P \tag{2,3}$$

The boundary conditions at the singularity are given again, as in I by

$$\left.\begin{aligned} \int D_\nu \, ds &= \eta \quad ; \quad \int B_\nu \, ds = 0 \\ \int H_s \, ds &= -\eta v \quad ; \quad \int E_s \, ds = 0 \end{aligned}\right\} \tag{2,4}$$

3. Solution in the case of Zero-approximation.

Taking $\rho = a$ the equations (2,1) can be shown, in consonance with the boundary conditions (2,4), to be satisfied by

$$\left.\begin{aligned} D_\rho &= \frac{\eta}{2\pi r} \cdot \frac{\rho - a}{r} \\ D_\chi &= \frac{\eta}{2\pi r} \cdot \frac{\chi}{r} \end{aligned}\right\} \tag{3,1}$$

and,

$$\left.\begin{aligned} B_\rho &= \lambda E_\chi \quad ; \quad B_\chi = -\lambda E_\rho \\ H_\rho &= \lambda D_\chi \quad ; \quad H_\chi = -\lambda D_\rho \end{aligned}\right\} \tag{3,2}$$

where $\lambda = v$. Also the above two sets of equations give $(\vec{B}\vec{E}) = 0$; $(\vec{D}\vec{H}) = 0$, showing that we are justified in using the several action-functions mentioned in the introduction.

From (3,1) and (3,2) we can easily express E_ρ, E_χ

in terms of D_ρ and D_χ. Introducing the absolute field-constant b we can write

$$H = \frac{1}{2} b^2 \log\left(1 + \frac{\vec{D}^2 - \vec{H}^2}{b^2}\right),$$ and use the relations

$$\vec{E} = \frac{\partial H}{\partial \vec{D}} \; ; \; \vec{B} = \frac{\partial H}{\partial \vec{H}}. \qquad (3,3)$$

That relations (3,3) also hold good in the several forms of the action functions given by Infeld can be easily seen from the relations

$$\left.\begin{aligned} T &= L + H \\ p^{kl} &= \partial T / \partial f_{kl} \\ f^{*kl} &= \partial T / \partial p^{*}_{kl} \end{aligned}\right\}$$

We can therefore obtain

$$\left.\begin{aligned} E_\rho &= \frac{\partial H}{\partial D_\rho} = \frac{D_\rho}{1 + \frac{\alpha^2}{b^2}(D_\rho^2 + D_\chi^2)} \\ E_\chi &= \frac{\partial H}{\partial D_\chi} = \frac{D_\chi}{1 + \frac{\alpha^2}{b^2}(D_\rho^2 + D_\chi^2)} \end{aligned}\right\}$$

or, introducing the values of D_ρ and D_χ from (3,1)

$$\left.\begin{aligned} E_\rho &= \frac{\eta}{2\pi} \cdot \frac{\rho - a}{r^2 + r_1^2} \\ E_\chi &= \frac{\eta}{2\pi} \cdot \frac{z}{r^2 + r_1^2} \end{aligned}\right\} \qquad (3,4)$$

(3,2) then immediately leads to

$$\left.\begin{aligned} H_\rho &= \frac{\eta v}{2\pi r}\cdot\frac{z}{r} \\ H_z &= -\frac{\eta v}{2\pi r}\cdot\frac{\rho-a}{r} \end{aligned}\right\} \qquad (3.5)$$

and,

$$\left.\begin{aligned} B_\rho &= \frac{\eta v}{2\pi}\cdot\frac{z}{r^2+r_1^2} \\ B_z &= -\frac{\eta v}{2\pi}\cdot\frac{\rho-a}{r^2+r_1^2} \end{aligned}\right\} \qquad (3,6)$$

(3,1), (3,4) - (3,6) constitute the field theory solutions in the case of Zero-approximation.

4. Calculation of the energy.

We adopt here the same procedure outlined in I, for the calculation of energy and angular momentum and hence there is no need to alter the contributions to these two quantities arising from the classical part. The energy-momentum tensor is given by[8],

$$T_k^{\ l} = \frac{1}{2}\left\{ T\delta_k^l + \left(p^{lm} f_{mk} + f^{*lm} p^*_{mk} \right) \right\} \qquad (4,1)$$

This gives

$$T_4^4 = \frac{1}{2}\left\{ T + (\vec{D}\,\vec{E}) + (\vec{H}\,\vec{B}) \right\} \qquad (4,2)$$

The action function of Hoffmann and Infeld is given by[9]

(8) See II, p.769; equation (3,4)

(9) Ibid, p.768, equation (1,20) - There is an obvious misprint here; should be

$$T = \epsilon - \log \epsilon - 1 \qquad (4,3)$$

with $\epsilon = \left(-F/P\right)^{1/2}$ so that, using (2,3) we can write (introducing b)

$$T/b^2 = \frac{1}{1+P} + \log\left(1+P\right) - 1 ;$$

From (3,1) and (3,5)

$$P = \frac{1}{b^2}\left(\vec{D}^2 - \vec{H}^2\right) = \frac{r_1^2}{r^2}, \text{ and } T \text{ reduces to}$$

$$T = b^2 \left\{ \log\left(1 + \frac{r_1^2}{r^2}\right) - \frac{r_1^2}{r^2 + r_1^2} \right\}. \qquad (4,4)$$

From (3,1) and (3,4)

$$(\vec{D}\,\vec{E}) = \frac{\eta^2}{4\pi^2} \cdot \frac{1}{r^2 + r_1^2} \qquad (4,5).$$

and from (3,5) and (3,6)

$$(\vec{B}\,\vec{H}) = \frac{\eta^2 v^2}{4\pi^2} \cdot \frac{1}{r^2 + r_1^2} \qquad (4,6)$$

Substituting from (4,4), (4,5) and (4,6) in (4,2) the T_4^4-component is given by

$$U_f = T_4^4 = \frac{1}{2} b^2 \left\{ \log\left(1 + \frac{r_1^2}{r^2}\right) - \frac{r_1^2}{r^2 + r_1^2} \right\} + \frac{\eta^2\beta^2}{8\pi^2} \cdot \frac{1}{r^2 + r_1^2}$$

$$E_f = \int U_f \, dV = 4\pi^2 a \int_0^r U_f \, r\, dr$$

$$= 2\pi^2 a b^2 \int_0^r r \log\left(1 + \frac{r_1^2}{r^2}\right) dr + 2\pi^2 a b^2 r_1^2 \left(\frac{\beta^2}{\alpha^2} - 1\right) \int_0^r \frac{r\,dr}{r^2 + r_1^2}$$

$$= 2\pi^2 a b^2 \int_0^r r \log\left(1 + \frac{r_1^2}{r^2}\right) dr + a\eta^2 v^2 \int_0^r \frac{r\,dr}{r^2 + r_1^2}.$$

$$= \frac{1}{2} a\eta^2\alpha^2 \int_0^x x \log\left(1+\frac{1}{x^2}\right) dx + \frac{1}{2} a\eta^2 v^2 \log(1+x^2)$$

putting $r = r_1 x$

$$= \frac{1}{2} a\eta^2\alpha^2 \left[\left\{\frac{1}{2} x^2 \log\left(\frac{1+x^2}{x^2}\right)\right\} + \frac{1}{2}\log(1+x^2) - x^2\log x\right]$$
$$+ \frac{1}{2} a\eta^2 v^2 \log(1+x^2)$$

$$= \frac{1}{4} a\eta^2\alpha^2 \left\{ x^2 \log\left(\frac{1+x^2}{x^2}\right)\right\} + \frac{1}{4} a\eta^2\beta^2 \log(1+x^2) \qquad (4,7)$$

In the field theory expressions we assume that r is large compared with r_1 i.e., that $x = r/r_1$ is large and hence the expression within brackets in the first term of (4,7) is of the order unity, as can be seen from the fact that

$$\underset{x\to\infty}{Lt}\; x^2\log\left(1+\frac{1}{x^2}\right) = \underset{y\to 0}{Lt}\; \frac{\log(1+y^2)}{y^2} = \underset{y\to 0}{Lt}\; \frac{2y/(1+y^2)}{2y} = 1$$

Further α can be neglected since we assume that the velocity v is very nearly equal to the velocity of light, here taken equal to unity. We can therefore retain only the second term, wherein we can replace $(1+x^2)$ by x^2, β by 1, and finally obtain

$$E_f = a\eta^2 \log(r/r_1) \qquad (4,8)$$

From equation (7,4), p.369 of I

$$E_{el} = a\eta^2 \log(8a/r) \qquad (4,9)$$

Combining (4,8) and (4,9) we get for the total energy

$$E = E_f + E_{cl} = a\eta^2 \log (8a/r_1) \qquad (4,10)$$

again a finite value of exactly the same form as in I but only differing from it in having $8a$ in place of $16a$.

5. <u>Calculation of the angular-momentum</u>.

Confining ourselves as above to the field-theory part of the angular momentum, we can write

$$\int M_f \, dV = \int \rho \left(D_z B_\rho - B_z D_\rho \right) dV$$

$$= \frac{\eta^2 v}{4\pi^2} \iint \frac{\rho \cdot 2\pi\rho \, d\rho \, dz}{r^2 + r_1^2}, \quad \text{using } (3,1), (3,6)$$

$$\doteq \frac{\eta^2 v}{2\pi} \int_0^r \int_0^{2\pi} \frac{(a + r\cos\phi)^2 \, r \, dr \, d\phi}{r^2 + r_1^2}$$

$$= \eta^2 v a^2 \int_0^r \frac{r \, dr}{r^2 + r_1^2} + \frac{1}{2} \eta^2 v \int_0^r \frac{r^3 \, dr}{r^2 + r_1^2}$$

$$= \frac{1}{2} \eta^2 v \left(a^2 - \frac{1}{2} r_1^2 \right) \log \left(1 + \frac{r_1^2}{r^2} \right) + \frac{1}{4} \eta^2 v r^2$$

$$= \eta^2 v a^2 \log \left(\frac{r}{r_1} \right) + \frac{1}{4} \eta^2 v r^2, \text{ neglecting term in } r_1^2.$$

From I, p-372, equation (8,4)

$$\int M_{cl} \, dV = \eta^2 v a^2 \log \left(\frac{8a}{r} \right) - \frac{1}{4} \eta^2 v r^2; \text{ therefore}$$

$$M = a^2 \eta^2 v \log (8a/r_1) \qquad (5)$$

giving a finite value for the angular momentum of the same form as in I.

6. Infeld's action-functions.

The several types of action-functions proposed by Infeld are included under the one-parameter group

$$T = \frac{1}{\epsilon} + (1-\gamma)\log\epsilon - (1+\gamma) + \gamma\epsilon \qquad (6,1)$$

γ being the parameter. The relation connecting F and P is given by using $T_\epsilon = -P$, or $\epsilon^2 T_\epsilon = F$ as

$$-\frac{1}{\epsilon^2} + \frac{1-\gamma}{\epsilon} + \gamma = -F \qquad (6,2)$$

Solving (6,2) as a quadratic in $(1/\epsilon)$ we get

$$2/\epsilon = (1-\gamma) \pm \sqrt{(1+\gamma)^2 + 4P}$$

The choice of the sign can be made by going to the limiting case of weak fields i.e. $P \to 0$; in this case

$$2/\epsilon \to 2, \text{ or } -2\gamma; \text{ ie } \epsilon \to 1, \text{ or } -1/\gamma$$

according as the $+$ or $-$ sign is taken; but in this case of limiting weak fields $\epsilon \to 1$ and $\gamma \to \infty$ we therefore take the positive sign of the quare root and get

$$2/\epsilon = (1-\gamma) + \sqrt{(1+\gamma)^2 + 4P} \qquad (6,3)$$

The Hamiltonian H is given by

$$H = \frac{1}{2}(T - R) \qquad (6,4)$$

T is a function of ϵ and $R = -\epsilon P$. Further from (6,3) ϵ is a function of P so that we can get H as a function of P only.

From (3,3),

$$\vec{E} = \frac{\partial H}{\partial \vec{D}} = \frac{\partial H}{\partial P}\cdot\frac{\partial P}{\partial \vec{D}} = 2\vec{D}\frac{\partial H}{\partial P}$$

$$= \vec{D}\left(\frac{\partial T}{\partial P} - \frac{\partial R}{\partial P}\right)$$

$$= \vec{D}\left(\frac{\partial T}{\partial \epsilon}\cdot\frac{\partial \epsilon}{\partial P} + \epsilon + P\frac{\partial \epsilon}{\partial P}\right) \qquad (\because R = -\epsilon P)$$

$$= \epsilon\vec{D}, \qquad (\text{using } T_\epsilon = -P) \tag{6,5}$$

Similarly

$$\vec{B} = -\epsilon\vec{H} \tag{6,6}$$

From (3,1), (6,5) and (6,3) we can find the field theory expressions for E_ρ and E_χ. In fact

$$P = (\vec{D}^2 - \vec{H}^2) = \alpha^2\vec{D}^2 = \eta^2\alpha^2/4\pi^2r^2 \quad \text{and}$$

$$\epsilon = \frac{2}{(1-\gamma)+\sqrt{(1+\gamma)^2+\eta^2\alpha^2/\pi^2r^2}} = \frac{2}{(1-\gamma)+\sqrt{(1+\gamma)^2+4\gamma_1^2/r^2}} \tag{6,3a}$$

Hence from (6,5)

$$\left.\begin{aligned} E_\rho &= \frac{2D_\rho}{(1-\gamma)+\sqrt{(1+\gamma)^2+4\gamma_1^2/r^2}} = \frac{\eta}{\pi r^2}\cdot\frac{\rho-a}{(1-\gamma)+\sqrt{(1+\gamma)^2+4\gamma_1^2/r^2}} \\ E_\chi &= \frac{2D_\chi}{(1-\gamma)+\sqrt{(1+\gamma)^2+4\gamma_1^2/r^2}} = \frac{\eta}{\pi r^2}\cdot\frac{\chi}{(1-\gamma)+\sqrt{(1+\gamma)^2+4\gamma_1^2/r^2}} \end{aligned}\right\}$$

Similar expressions could be obtained from (3,2) for B_ρ, B_χ and H_ρ, H_χ. The T_4^4-component of the energy impulse tensor is given by (4,2) and can be expressed in terms of r using (6.3a) and the above expressions given by the field theory solutions.

$$T_4^4 = \frac{1}{2}\left\{T + (\vec{D}\vec{E}) + (\vec{B}\vec{H})\right\}$$

$$= \frac{1}{2}\left\{T + \epsilon^2\vec{D}^2 + \lambda^2\epsilon^2\vec{D}^2\right\}$$

$$= \frac{1}{2}\left\{T + \beta^2\epsilon^2\vec{D}^2\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{\epsilon} + (1-\gamma)\log\epsilon + \gamma\epsilon - (1+\gamma) + \frac{\beta^2\eta^2\epsilon^2}{4\pi^2 r^2}\right\} \qquad (6,7)$$

Where ϵ is given as a function of r from (6.3a). We can proceed to calculate the field theory part of the energy as before. The calculations are a bit tedious and the final result, after the necessary approximations of (r/r_1) being large, $\alpha \to 0, \beta \to 2$ is, for the case $\gamma \geq 0$, given by

$$E_f = a\eta^2 \log\left\{\frac{(1+\gamma)r}{r_1}\right\} \qquad (6,8)$$

leading to the total energy

$$E = a\eta^2 \log\left\{\frac{\delta a(1+\gamma)}{r_1}\right\} \qquad (6,9)$$

again giving a finite value. This includes the Born-case and the case $\gamma = 0$ of Infeld which is the counter-

of the Hoffmann-Infeld action function as particular cases given by $\gamma = 1$ and $\gamma = 0$ It is practically certain that the calculations with the angular - momentum also lead to the same results as in I.

7. Conclusion.

From the expressions for energy and angular-momentum derived above it is clear that all the considerations of our previous investigation are exactly reproduced. The main purpose of this was to test out the possibility of explaining spin classically as suggested by Kramers. In trying to interpret the results of Kramers on the basis of the unitary field theory Born used the point singularity model, and although it was possible to construct a formal theory there arose contradictions which suggested that point-singularities may not be the correct representations of the elementary particles. The result of the work on the ring-singularity shows that this too does not give a correct representation. The alternative suggests itself that one had better try other types of singularities, but the complications that would be inherent in assuming such higher singularities and the consequent mathematical difficulties almost

force us to the conclusion that it is perhaps better to go back to the point singularity and examine its possibilities(10) more thoroughly than has been done so far.

---

(10) Prof. Born has kindly informed me by letter that he has examined this question carefully in the course of his lectures to the Henri Poincaré Institute, Paris (to be shortly published) and come to the conclusion that point-singularities alone constitute the correct representation.

P.T.O.

PART B.

Ring-Singularity in the general case.

1. Introduction.

In the two previous papers we have assumed the singular line to be a circle, and that the current and charge are uniformly distributed on it. Also the investigation was carried out with the aid of cylindrical coordinates because of the assumptions made. We shall give up these assumptions in this paper and take the singular line to be any arbitrary closed curve with arbitrary distribution of current and density on it. Without considering the question of the equations of motion in an external field, we shall confine ourselves to the chief question of "self-force;" and show that this vanishes in a point of the line-singularity in virtue of the minimum principle. It is in this very important respect that Born's theory differs from the old Maxwell-Lorentz theory wherein the question of self-force is connected with the difficult question of structure of the electron. We shall also derive the conservation laws with zero on their right hand sides. The method adopted will be that given by Born in his two papers[**] on the

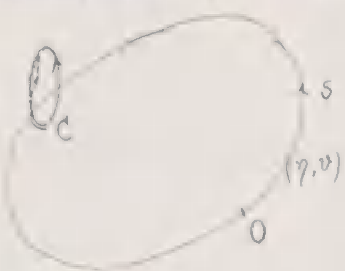
unitary theory of field and matter.

2. Variation principle and field equations.

The Lagrangian of the field will be denoted by $L(f_{kl})$ and we will not assume any special form of L as done in the previous investigations. To this Lagrangian of the field we add a Lagrangian of the singularity which we suppose to have the form $\ell(\phi_k)\delta$ where δ is a symbolic function of the type introduced by Dirac. We assume that $\delta(x,y,z)=0$ at every point except points on the singular line whose equation can be taken as

$$x_0 = x_0(s)\,;\quad y_0 = y_0(s)\,;\quad z_0 = z_0(s) \tag{2,1}$$

S being the arc length measured on the line from a fixed point O on it. δ is infinite at (x_0, y_0, z_0) in such a way that



$$\int \delta \, d\sigma = 1 \tag{2,2}$$

where $d\sigma$ is a surface element normal to ds

Since the absolute value of the potential in the singularity has a definite meaning, the introduction of ϕ_k in ℓ is permissible. The variation principle

governing field and matter is

$$\int\left\{L(f_{kl}) + l(\phi_k)\delta\right\} dv\,dt = Extremum \tag{2,3}$$

where the volume element dv can be taken $dv = d\sigma ds$; we define the second kind of field components in the usual way by

$$p^{kl} = \frac{\partial L}{\partial f_{kl}} \tag{2,4}$$

and further put

$$\eta^k = \frac{\partial l}{\partial \phi_k} \tag{2,5}$$

As the f_{kl} are connected with the potentials ϕ_k by

$$f_{kl} = \frac{\partial \phi_k}{\partial x^l} - \frac{\partial \phi_l}{\partial x^k},$$

one has the identities

$$\frac{\partial f^{*kl}}{\partial x^l} = 0, \text{ or } \frac{\partial f_{kl}}{\partial x^m} + \frac{\partial f_{lm}}{\partial x^k} + \frac{\partial f_{mk}}{\partial x^l} = 0,$$

which in the space-vector notation becomes

$$\left.\begin{aligned} rot\,\vec{E} + \dot{\vec{B}} &= 0 \\ div\,\vec{B} &= 0 \end{aligned}\right\} \tag{2,6}$$

The Eulerian equations of the variation principle are

$$\frac{\partial p^{kl}}{\partial x^l} = \eta^k \delta \tag{2,7}$$

or, in the space-vector notation with

$$(\eta^1, \eta^2, \eta^3, \eta^4) \longrightarrow (\eta \vec{v}, \eta)$$

$$\left.\begin{aligned} \operatorname{rot} \vec{H} - \dot{\vec{D}} &= \eta \vec{v} \delta \\ \operatorname{div} \vec{D} &= \eta \delta \end{aligned}\right\} \qquad (2,8)$$

Here η and $\vec{v}$ are functions of s and t, and not consyants as in the previous investigations. From (2,5) it is easy to see that η denotes the linear charge density. If ρ be the space density we have the total charge e given by

$$e = \int \rho \, dv = \iint \rho \, d\sigma \, ds$$

also, if $e = \int \eta \, ds$, comparing these forms and using (2,2) we can write $\rho = \eta \delta$, showing in virtue of dimensions that η is a linear density.

3. Equation of continuity.

From (2,7) it follows, on account of the antisymmetry of the tensor p^{kl}

$$\frac{\partial \eta^k \delta}{\partial x^k} = 0 \qquad (3,1)$$

This becomes in space-vector notation

$$\frac{\partial (\eta \delta)}{\partial t} + \operatorname{div} (\eta \vec{v} \delta) = 0 \, .$$

Integrating this over space with the volume element $dv = d\sigma\, ds$,

$$\iint \left\{ \frac{\partial(\eta\delta)}{\partial t} + \operatorname{div}(\eta\vec{v}\delta) \right\} d\sigma\, ds = 0,$$

ie $\int \frac{\partial\eta}{\partial t} ds + \iint \operatorname{div}(\eta\vec{v}\delta)\, d\sigma\, ds = 0$; Also, $\operatorname{div}(\eta\vec{v}\delta) = \nabla\cdot(\eta\vec{v}\delta) = \frac{\partial}{\partial s}(\eta v\delta)$, since $\vec{v}$ has the direction of the tangent to the singular line.

Hence $\iint \operatorname{div}(\eta\vec{v}\delta)\, d\sigma\, ds = \int ds \frac{\partial}{\partial s}(\eta v) \int \delta\, d\sigma$

since δ is in dependent of s. Thus we have

$$\int \left\{ \frac{\partial\eta}{\partial t} + \frac{\partial}{\partial s}(\eta v) \right\} ds = 0,$$

and this independently of any dependence of η and v on S. Hence

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial s}(\eta v) = 0 \qquad (3,2)$$

which is theequation of continuity.

4. Boundary conditions at the singularity.

Consider an infinitesimal torus surface surrounding the singular line the cross-section of this surface being a curve C (See Fig). Let dl be a line-element of curve C, so that a surface element of this torus-

surface can be taken as $ds \cdot dl$. $d\sigma$ is, as before, the area of cross-section normal to ds. The equations (2,6), (2,8) are equivalent to the postulate, that the corresponding homogeneous equations

$$\left.\begin{array}{ll} \mathrm{rot}\,\vec{H} - \dot{\vec{D}} = 0 \; ; & \mathrm{rot}\,\vec{E} + \dot{\vec{B}} = 0 \\ \mathrm{div}\,\vec{D} = 0 \; ; & \mathrm{div}\,\vec{B} = 0 \end{array}\right\} \qquad (4,1)$$

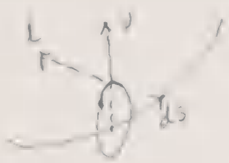
hold at any point except at points on the singular line where certain boundary conditions have to be fulfilled. These are found by integrating the differential equations (2,6), (2,8) over the small torus-surface. Integrating

$$\mathrm{div}\,\vec{D} = \eta\delta$$

over the volume of the torus-surface

$$\iint_0 \mathrm{div}\,\vec{D}\, d\sigma\, ds = \iint_0 \eta\delta\, d\sigma\, ds = \int \eta\, ds = e \qquad (4,2)$$

and the left-hand side can be transformed to the surface-integral $\iint D_\nu\, ds\, dl$.



Hence comparing with (4,2) we get

$$\left.\begin{array}{l} \int_0 D_\nu\, dl = \eta \\ \text{similarly} \quad \int_0 B_\nu\, dl = 0 \end{array}\right\} \qquad (4,3)$$

Again from $\mathrm{rot}\,\vec{H} - \dot{\vec{D}} = \eta\vec{v}\delta$

$$\iint \operatorname{rot} \vec{H} \, d\sigma \, ds = \iint \eta \vec{v} \delta \, d\sigma \, ds,$$ since

it is the essence of the unitary theory that

$$\int \dot{\vec{D}} \, dv = 0;$$

The right-hand side gives

$\int \eta \vec{v} \, ds$, and the left-hande isequal to

$$\iint_0 (\vec{n} \times \vec{H}) \, ds \, dl,$$

where $\vec{n}$ is the normal to the surface element. Hence

$$\left.\begin{aligned} &\int_0 (\vec{n} \times \vec{H}) \, dl = \eta \vec{v} \\ \text{Similarly, } &\int_0 (\vec{n} \times \vec{B}) \, dl = 0 \end{aligned}\right\} \qquad (4,4)$$

(4,3) - (4,4) are the boundary conditions at the singularity expressed as line-integrals around the singular line.

5. Differential conservation laws.

The energy-impulse tensor is given, in the usual way, by

$$T_k^{\ l} = L\delta_k^l - p^{ln} f_{kn} \qquad (5,1)$$

and the differential conservation-laws by

$$\frac{\partial T_k^{\ l}}{\partial x^l} = \delta f_{km} \eta^m; \qquad (5,2)$$

writing out (5,2) in the space-vector notation

$$\left.\begin{aligned} &\frac{\partial g_x}{\partial t} + \operatorname{div} \vec{X} = \delta\eta \left\{ \vec{E} + (\vec{v} \times \vec{B}) \right\}_x \\ &\text{-----------------------} \\ &\text{-----------------------} \\ &\frac{\partial U}{\partial t} + \operatorname{div} \vec{S} = -\delta\eta (\vec{v} \cdot \vec{E}) \end{aligned}\right\} \quad (5,3)$$

Integrating (5,3) over the small torus-surface surrounding the singular line, we get, (using $\int \delta\, d\sigma = 1$),

$$\left.\begin{aligned} &\iint_0 (\vec{X} \cdot \vec{n})\, ds\, dl = \int \eta \left\{ \vec{E}_0 + (\vec{v} \times \vec{B}_0) \right\}_x ds \\ &\text{-----------------------} \\ &\iint_0 (\vec{S} \cdot \vec{n})\, ds\, dl = -\int \eta (\vec{v} \cdot \vec{E}_0)\, ds \end{aligned}\right\} \quad (5,4)$$

the subscript 0 denoting values at any point of the singular line.

The right hand sides of (5,4) are the expressions for the "self-force" expressed as line-integrals extended over the singular-line. We will now proceed to show that these expressions vanish in virtue of the variation principle for varying motions of the singularity.

6. Dynamical boundary conditions at the singularity.

A variation of the motion of the singularity consists in the variation of $\vec{r}_0(s,t)$, the components of $\vec{r}_0$ being $\left(x_0(s), y_0(s), z_0(s) \right)$, the variation being however independent of s. We can assume that l has the form

since this gives the same field-equations (2,6),(2,8). Here $\phi, \vec{A}$ are to be considered as functions of $\vec{r}_0(s,t)$ whereas $\vec{v} = \dot{\vec{r}}_0(s,t)$. In view of the fact that the potentials $\phi, \vec{A}$ are functions having singular points at $\vec{r}_0(s)$, the variation will have to be carried out in the elaborate way indicated by Born in the two papers mentioned above. As done therein we introduce instead of x, y, z, t the parameters ξ, η, ζ, τ, and write the variation principle as

$$\int d\tau \int d\xi\, d\eta\, d\zeta \left\{ \dot{t}_0 (\rho_0 \phi) - \dot{\vec{r}}_0 \rho_0 \vec{A} \right\} = \text{Extremum} \qquad (6,2)$$

with the subsidiary condition

$$\dot{t}_0^{\,2} - \dot{x}_0^{\,2} - \dot{y}_0^{\,2} - \dot{z}_0^{\,2} = 1 \qquad (6,3)$$

The variation principle (6,2) is the well-known principle giving Lorentz-equations of motions, and can

be treated in exactly the same way. We take account of the subsidiary condition (6,3) by a Lagrangian multiplier μ, and get

$$\left.\begin{aligned} \iiint d\xi\, d\eta\, d\zeta \left(\frac{d\mu \dot{\vec{r}}_0}{d\tau} - \vec{K} \right) &= 0 \\ \iiint d\xi\, d\eta\, d\zeta \left(\frac{d\mu \dot{t}_0}{d\tau} - K \right) &= 0 \end{aligned}\right\} \qquad (6,4)$$

where
$$\left.\begin{aligned}\vec{K} &= \dot{t}_0 \rho_0 \vec{E} + \rho_0 (\dot{\vec{r}}_0 \times \vec{B}) \\ k &= \rho_0 (\dot{\vec{r}}_0 \vec{E})\end{aligned}\right\} \tag{6,5}$$

Going to the limiting process of contracting the current to a world line of any point of the singular line and also taking $\mu = 0$, in consonance with the ideas of the unitary theory, we get $(\rho_0 \to \eta\delta)$
$$\left.\begin{aligned}\iiint dv \cdot \eta\delta \left\{ \vec{E} + (\vec{v} \times \vec{B}) \right\} &= 0 \\ \iiint dv \cdot \eta\delta (\vec{v}\vec{E}) &= 0\end{aligned}\right\}$$

ie.
$$\left.\begin{aligned}\int \eta \left\{ \vec{E}_0 + (\vec{v} \times \vec{B}_0) \right\} ds &= 0 \\ -\int \eta (\vec{v}\vec{E}_0)\, ds &= 0\end{aligned}\right\} \tag{6,6}$$

which show that the "self-force" vanishes. The left-hand sides of (6,6) are exactly the same as the right-hand sides of (5,4).

We can now replace (5,3) by the equations
$$\left.\begin{aligned}\partial S_x/\partial t + \operatorname{div} \vec{X} &= 0, \ldots \\ \partial U/\partial t + \operatorname{div} \vec{S} &= 0\end{aligned}\right\} \tag{6,7}$$

which hold for all fields satisfying
$$\left.\begin{aligned}\iint_0 (\vec{X} \cdot \vec{n})\, dl\, ds &= 0 \\ \iint_0 (\vec{S} \cdot \vec{n})\, dl\, ds &= 0\end{aligned}\right\} \tag{6,8}$$

Let us now define the total energy and momentum by

$$E = \int U dv \, ; \quad \vec{G} = \int \vec{S} dv \tag{6,9}$$

Integrating (6,7) over the whole of space excluding the singular line by a small torus surface, we get

$$\left.\begin{aligned} \dot{G}_x + \int_\infty (\vec{X}\cdot\vec{n})\, dl\, ds = 0 \\ \dot{E} + \int_\infty (\vec{S}\cdot\vec{n})\, dl\, ds = 0 \end{aligned}\right\} \tag{6,10}$$

since, in consequence of (6,8), the integrals over the infinitesimal torus-surface vanish. (6,10) is standard form of the conservation laws with Zero on the right-hand sides.

7. Nature of the singular line.

We now proceed to show that the condition of the vanishing of the self-force leads to the conclusion that the density and current density are independent of S, and that the singular line is a circle.

Equating to Zero the components of the self-force along the tangent, principal normal and binormal (S, p, b directions)

$$\int \eta E_{0s}\, ds = 0; \int \eta \left(E_{0p} - v B_{0b}\right) ds = 0; \quad \int \eta \left(E_{0b} + v B_{0p}\right) ds = 0$$

and $\int \eta v E_{0s}\, ds = 0.$

Of these, we consider the equations

$$\left.\begin{aligned} \int \eta E_{0s}\, ds &= 0 \\ \int \eta v E_{0s}\, ds &= 0 \\ \text{along with,} \quad \int \eta\, ds &= e \end{aligned}\right\} \tag{7,1}$$

Introducing the parameter $u = (2\pi/\mathcal{L})s$, where $\mathcal{L}$ = perimeter of the curve, we can take the Fourier-expansion of E_{0s} as

$$E_{0s} = \frac{1}{2}\alpha_0 + \sum \alpha_k \cos ku + \sum \beta_k \sin ku \tag{7,2}$$

From the first of the conditions (7,1) we have

$$\int \frac{1}{2}\alpha_0 \eta\, du + \sum \alpha_k \int \eta \cos ku\, du + \sum \beta_k \int \eta \sin ku\, du = 0, \tag{7,3}$$

Here all the $\alpha's$, $\beta's$ are entirely arbitrary, but the third condition of (7,1) shows that in order that (7,3) may hold

$$\alpha_0 = 0$$

$$\text{and,} \int \eta \cos ku\, du = 0, \int \eta \sin ku\, du = 0, \quad (\text{for all } k)$$

$$\text{but} \int \eta\, du \neq 0, \tag{7,4}$$

Thus all the trigonometric moments are zero, with however (7,4); hence η is a constant independent of s. In an exactly similar manner the second

condition of (7,1) shows that ηv is a constant. Therefore v is a constant and the singular line a circle. Also from the equation of continuity

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial s}(\eta v) = 0,$$

we get $\frac{\partial \eta}{\partial t} = 0$, i.e. η is also independent of t.

8. Conclusion.

It would be possible, next, to consider the equations of motion of the ring-singularity in an external field which is "constant" over the diameter of the ring, just as is done in the papers of Born for the point singularity. But in view of the negative results we have obtained in the stationary case with the circular ring model, it does not seem profitable to carry out this investigation any further. Enough work has been done to bring out the fundamental principles of the classical part of Born's theory as applied to the ring-singularity.

----

(3)

ON THE FINE - STRUCTURE OF BALMER LINES.

Reprinted from "The Proceedings of the Indian Academy of Sciences", Vol. V, No. 1, Sec. A, 1937.

ON THE FINE STRUCTURE OF THE BALMER LINES.

BY B. S. MADHAVA RAO.

(University of Mysore)

(From the Department of Physics, Indian Institute of Science, Bangalore.)

Received January 12, 1937.

1. *Introduction.*

SOME recent experiments[1] on the fine structure of the Balmer lines of Hydrogen have shown a disagreement between calculated and observed term values. Other experiments[2] reach the opposite conclusion that no such disagreement exists. Houston and Hsieh state that the order of the magnitude of the effect observed by them is α times the fine-structure separation (α = fine-structure constant), and that the theory of fine structure requires modification. They have further made the suggestion (pointed out by Bohr and Oppenheimer) that the discrepancy might perhaps be corrected by taking into account the interaction between the electron and the radiation field due to the transitions between several fine-structure levels.

Several attempts have been made to sharpen the theory of the fine structure so as to enable an explanation of this discrepancy. Heller and Motz[3] have attempted to replace the Coulomb potential of the nuclear field by the potential of a static field given by the new field theory of Born and Infeld.[4] Treating the difference between the Born and Coulomb potentials as a perturbation, they calculate the term shifts in the one electron Schrödinger problem and conclude that the corrections obtained, although in the right direction, are much too small to explain the observed discrepancy. They further state that a rigorous treatment with the Dirac eigenfunctions does not materially alter the situation. Another attempt using the Born potential has been made by Meixner.[5] In place of the Schrödinger wave equation employed by Heller and Motz, he uses the Pauli equation which takes account of spin and also the relativistic correction. Treating $\phi_B - \phi_C$ as a

[1] W. V. Houston and Y. M. Hsieh, *Phys. Rev.*, 1934, **45**, 263.
R. C. Williams and R. C. Gibbs, *ibid.*, p. 475.

[2] F. H. Spedding, C. D. Shane and N. S. Grace, *Phys. Rev.*, 1935, **47**, 38.

[3] G. Heller and L. Motz, *Phys. Rev.*, 1934, **46**, 202.

[4] M. Born and L. Infeld, *Proc. Roy. Soc.*, 1934, **A144**, 425.

[5] J. Meixner, *Ann. der Phys.*, 1935, [5] **23**, 371.

perturbation the result is obtained that the spin and relativity corrections are half as large as those obtained from Schrödinger's theory. Nevertheless, the total correction is shown to be negligible if one uses for the electron the radius of Born's field theory. If, instead of this radius, the radius as modified by Born and Schrödinger[6] be used the result, as shown by Meixner, is that nearly 30% of the discrepancy can be explained.

I have attempted in this paper to estimate the order of magnitude of the corrections obtained by taking into account the interaction of the electron with its own radiation field. For this purpose, I have used Born's field theory since by its unitary nature this theory can be expected to take care of this interaction automatically. This has been made possible by the two recent papers of Born[7] on the unitary theory of field and matter wherein the effect of an external field is taken into consideration provided that the external field be subject to the restriction that it be constant over the " diameter " of the electron. If we observe that the radius of the electron is of the order of 10^{-13} cm. and the radius of the H-atom 10^{-8} cm., it is evident that the external field of the nucleus satisfies the restriction stated above.

The method of procedure I adopt is to calculate the energy of the electromagnetic field wherein the electron is a singularity and is acted upon by the " constant " nuclear field, following the steps indicated in the above two papers of Born. The energy is obtained as the sum of two terms W_0 and ΔW, the former being the energy unperturbed by the external field and ΔW the perturbation energy. I next employ this perturbation term in the wave equation and estimate the correction in the several spectral terms.

The result I have obtained is that, if we use the Born radius of the electron, the corrections due to the interaction of the electron and the radiation field are negligible.[8] If, on the other hand, we use the Born-Schrödinger radius the corrections obtained explain one or two per cent. of the discrepancy. The notion of the Born-Schrödinger radius is subject, however, to great theoretical difficulties. For one thing it would spoil the

[6] M. Born and E. Schrödinger, *Nature*, 1935, **135**, 342.

[7] M. Born, *Proc. Ind. Acad. Sci.*, 1936, **3**, No. 1, 8; and *ibid.*, **3**, 2, 85.

[8] In a recent paper, Meixner (*Ann. der Phys.*, 1936, [5] **27**, 389) has treated the question of this correction by using the methods of Weisskopf and Wigner in Dirac's theory of radiation and has come to the same conclusion as mine. It is quite natural that this purely quantum treatment does not involve any question of radius of the electron. This paper deals comprehensively with the several possible theoretical aspects of the question.

beautiful coincidences in the phenomenon of scattering of light by light[9] on Born's classical field theory and Dirac's purely quantum hole theory. It would also contradict the value[10] $\pi e^4/mc^4$ (experimentally confirmed) of the cross-section for the scattering by electrons of light of long wave-length. If we, therefore, lay aside the Born-Schrödinger radius as untenable, we can conclude that the interaction of the electron and radiation field does not materially effect the energy levels. It is remarkable that this investigation and Meixner's totally different method should lead to identical conclusions.

2. *Energy of Electron in Constant External Field.*

We assume the field of the nucleus constant over the diameter of the electron and determine the energy of the latter following the method of Born in the two papers mentioned above. (See reference [7].)

The total field is written

$$\left.\begin{aligned} \vec{E} &= \vec{E}^e + \vec{E}^i\,; \quad \vec{B} = \vec{B}^e + \vec{B}^i \\ \text{and} \qquad \vec{D} &= \vec{D}^e + \vec{D}^i\,; \quad \vec{H} = \vec{H}^e + \vec{H}^i \end{aligned}\right\} \tag{1}$$

where the e- and i-fields denote the external field and field due to the electron itself. At infinity the fields $\vec{E}^i$, $\vec{B}^i$, $\vec{D}^i$ and $\vec{H}^i$ all tend to zero. The energy is calculated from

$$W = \int U\,dv \tag{2}$$

where U is the energy-density given by

$$U = \sqrt{1 + \vec{D}^2 + \vec{B}^2 + \vec{S}^2} - 1 \tag{3}$$

and

$$\vec{S} = (\vec{D} \times \vec{B}) = (\vec{E} \times \vec{H}).$$

We treat the e-fields as constants and also omit terms like $\vec{D}^{e^2}$ within the square root expression. This last is permissible since we can confine ourselves, so far as the range of integration is concerned, to a small neighbourhood of the singularity wherein almost all the energy is concentrated and wherein the e-fields are negligible compared to the i-fields. In the purely electrical case

$$U = \frac{b^2}{4\pi}\sqrt{1 + \frac{\vec{D}^2}{b^2}} - 1 \tag{4}$$

---

[9] See for *e.g.*, L. Euler, *Ann. der Phys.*, 1936, **26**, 398. Also M. Born, *Proc. Ind. Acad. Sci.*, 1935, **2**, No. 6, 533.

[10] See M. Born, *Naturw.*, 1932, **20**, 269; also, *Nature*, 1935, **136**, 952.

introducing the absolute field b and the conventional units. We assume the field $\vec{D}^e$ constant both in magnitude and direction and take the latter as the x-direction with the origin taken at the singularity. Thus we can take

$$\vec{D}^e \text{ as } (D^e = D_x^e,\ 0,\ 0)$$

and
$$\vec{D}^i \text{ as } \left(\frac{e}{r^2}\cdot\frac{x}{r},\ \frac{e}{r^2}\cdot\frac{y}{r},\ \frac{e}{r^2}\cdot\frac{z}{r}\right)$$

and put $\vec{D} = \vec{D}^e + \vec{D}^i$ in (4). This gives

$$W = \frac{b^2}{4\pi}\int_0^\infty\int_0^\pi \left(\sqrt{1 + \frac{D^{e2}}{b^2} + \frac{D^{i2}}{b^2} + \frac{2D^eD^i\cos\theta}{b^2}} - 1\right) 2\pi r^2 dr \sin\theta d\theta$$

θ being the angle between $\vec{D}^i$ and the x-direction.

$$\text{or,}\quad W = \frac{b^2}{4\pi}\int_0^\infty\int_0^\pi \left(\sqrt{1 + \frac{D^{i2}}{b^2} + \frac{2D^eD^i\cos\theta}{b^2}} - 1\right) 2\pi r^2 dr \sin\theta d\theta$$

$$= \frac{b^2}{4\pi}\int\int \left(\sqrt{1 + \frac{D^{i2}}{b^2}}\left\{1 + \frac{2D^eD^i\cos\theta/b^2}{1 + D^{i2}/b^2}\right\}^{\frac{1}{2}} - 1\right) 2\pi r^2 dr \sin\theta d\theta.$$

Expanding the square root with the aid of the Binomial theorem up to the second power of D^e we find in virtue of

$$\int_0^\pi \cos\theta \sin\theta d\theta = 0, \qquad \text{that}$$

$$W = W_0 - \frac{b^2}{8\pi}\int_0^\infty\int_0^\pi \frac{D^{e2}D^{i2}\cos^2\theta/b^4}{(1+D^{i2}/b^2)^{\frac{3}{2}}}\, 2\pi r^2 \sin\theta dr d\theta$$

$$= W_0 - \frac{D^{e2}}{6b^2}\int_0^\infty \frac{r^2 D^{i2}\, dr}{(1+D^{i2}/b^2)^{\frac{3}{2}}}$$

$$= W_0 - \frac{D^{e2}r_0^4}{6}\int_0^\infty \frac{r^4 dr}{(r^4+r_0^4)^{\frac{3}{2}}} \tag{5}$$

where
$$W_0 = \frac{b^2}{4\pi}\int\int \left(\sqrt{1 + \frac{D^{i2}}{b^2}} - 1\right) 2\pi r^2 dr \sin\theta d\theta$$

and we have made the substitutions $D^i = e/r^2$ and $b = e/r_0$, r_0 being the Born-radius[11] of the electron. The integral in (5) can be easily integrated

[11] See reference [4], p. 439.

by putting $r = r_0 x$ and is equal to

$$\frac{1}{r_0}\int_0^\infty \frac{x^4 dx}{(1+x^4)^{\frac{3}{2}}}$$

$$= -\frac{1}{2r_0}\int_0^\infty x\,\frac{d}{dx}\left(\frac{1}{\sqrt{1+x^4}}\right)dx$$

$$= \frac{1}{2r_0}\int_0^\infty \frac{dx}{\sqrt{1+x^4}} = \frac{f(0)}{2r_0}$$

where $f(0) - f(x) = \frac{1}{2}\,\mathrm{F}\left(\frac{1}{\sqrt{2}}, 2 \text{ arc tan } x\right)$ and F (k, 2 arc tan x) is the Jacobian elliptic integral of the first kind for $k = \frac{1}{\sqrt{2}}$.

Hence (5) reduces to

$$\mathrm{W} = \mathrm{W}_0 - \frac{\mathrm{D}^{e^2}\, r_0^3}{12} f(0). \tag{6}$$

Expression (6) shows how, on the present theory, the energy is modified when the effect of the external field is taken into consideration. We now put in (6),

$$\mathrm{D}^e = \frac{\mathrm{Z}e}{r^2} \tag{7}$$

the r being now referred to the nucleus as origin and obtain

$$\mathrm{W} = \mathrm{W}_0 - \frac{\mathrm{Z}^2 e^2 r_0^3 f(0)}{12}\cdot\frac{1}{r^4} = \mathrm{W}_0 - \frac{\lambda}{r^4}. \tag{8}$$

We now proceed to use the second term on the right hand-side of (8) as a perturbation and obtain the modification in the fine structure.

3. *Theory of Fine Structure.*

The present position of the theory of fine structure is that the correct explanation is afforded by Dirac's theory of the electron which gives results in accord with Sommerfeld's formula derived on the basis of the old quantum theory. The approximate two-matrix method of Pauli also gives correct results and is equivalent to the original Schrödinger-Gordon-Klein wave equation with the proper correction for spin. We can write the complete wave equation where the relativistic variation of mass and the spin are taken into consideration in the form[12]

$$\triangle\psi + \frac{8\pi^2 m}{h^2}\left(\mathrm{W}_0 + \frac{\mathrm{Z}e^2}{r} - \mathrm{V}_{\text{rel.}} - \mathrm{V}_{\text{spin}}\right)\psi = 0 \tag{9}$$

where $\mathrm{V}_{\text{rel.}}$ and V_{spin} can be treated as perturbation terms and are of the form

[12] See for *e.g.*, O. Halpern u. H. Thirring, *Erg. Exakt. Naturw.*, **Bd. 8**, S. 423,

$$V_{rel.} \sim \frac{1}{r^2} \text{ and } V_{spin} \sim \frac{1}{r^3}. \tag{10}$$

A treatment of equation (9) by using perturbation methods amounts to finding the averages $\overline{1/r^2}$ and $\overline{1/r^3}$ with respect to the radial wave functions of the unperturbed Schrödinger wave equation and gives the same result as obtained by Pauli's equation[13]

$$\left\{W + e\phi + \frac{\hbar^2}{2m}\Delta + \frac{1}{2mc^2}(W + e\phi)^2 + \frac{ie\hbar}{4m^2c^2}(\vec{E}\vec{p}) - \frac{e\hbar}{4m^2c^2}(\sigma[\vec{E} \times \vec{p}])\right\} \psi = 0 \tag{11}$$

where W = total energy, σ = Pauli's spin matrix, Δ = diograd, $\vec{p} = \frac{\hbar}{i}$ grad.

We now take account of a further perturbation due to our equation (8), wherein the W_0 is really to be taken equal to the expression within brackets in the left-hand side of (9), and introduce it in (9). The modified wave equation can be written as

$$\Delta\,\psi + \frac{8\pi^2 m}{h^2}\left(W_0 + \frac{Ze^2}{r} - V_{rel.} - V_{spin} - V_{field}\right)\psi = 0 \tag{12}$$

where $V_{field} \sim \frac{1}{r^4}$ is the perturbation term in (8) and can be taken to denote the interaction of electron and field.

Hence if ΔW be the energy correction we have

$$\Delta W = -\lambda\,\overline{1/r^4} \tag{13}$$

where the mean value is taken with respect to the radial Schrödinger function of the unperturbed equation.

Before proceeding to estimate the magnitude of the correction as shown by (13) it is necessary to remove a difficulty connected with $\overline{1/r^4}$ for the s-terms (*i.e.*, $l = 0$). A reference to Bethe's article p. 286 shows that this is equal to ∞, but we can get over the difficulty if we observe that for s-terms we are concerned with values of r nearly equal to zero and for such values we cannot neglect $e\phi = Ze^2/r$ as compared with the rest-energy $E_0 = mc^2$. This is the approximation made in deriving (11) from Dirac's exact wave equation. We have therefore to replace[14]

$$\overline{\frac{1}{r^4}} = \int R^2 \cdot \frac{1}{r^4} r^2 dr \text{ by } \int \frac{R^2 dr}{r^2\,(1 + e^2 z/mc^2 r)^2}$$

$$i.e., \text{ by } \int \frac{R^2 r^2 dr}{(r + \rho_0)^2} \qquad \text{where } \rho_0 = e^2/mc^2,\ z = 1 \tag{14}$$

and $R = R_{nl}$ is the radial Schrödinger function.

[13] See Bethe, *Handb. d. Phys. z. Auf.*, **Bd.** 24/1, p. 305.
[14] See Bethe, *ibid.*, footnote on p. 307.

Further in integrating (14) we can take the limits as 0 and ρ_0 themselves and use for R_{nl} in this region the value[15]

$$R_{nl} \approx \frac{1}{(2l+1)!} \sqrt{\frac{(n+l)!}{(n-l-1)!\,2n}} \left(\frac{2z}{n}\right)^{\frac{3}{2}+l} r^l \qquad (15)$$

An elementary calculation obtained by putting (15) in (14) and integrating between the limits 0 and ρ_0 shows that for *s*-terms no mistake would be committed as regards order of magnitude if $\overline{1/r^4}$ were to be put equal to unity multiplied by $Z^4/a_H{}^4$.

As regards higher terms like *p*- and *d*-terms the values of the numerical factors of $\overline{1/r^4}$ for $l = 1, 2$, etc., are small fractions which have all been calculated in Bethe's tables referred to above. Hence we can take the order of magnitude of the energy correction due to the interaction of the radiation and the electron as

$$\triangle W \sim \lambda Z^4/a_H{}^4 \sim Z^6 e^2 r_0{}^3 f(0)/12 a_H{}^4. \qquad (16)$$

With $Z = 1$, the correction in the wave number is given by

$$\frac{\triangle \nu_H}{c} \sim \frac{e^2 r_0{}^3\, f(0)}{12hc\, a_H{}^4} = \frac{\alpha r_0{}^3 f(0)}{2\pi \times 12\, a_H{}^4} \text{ cm.}^{-1} \qquad (17)$$

where α is the fine-structure constant given by $\alpha = 7 \cdot 28 \times 10^{-3}$. Putting in (17), $r_0 = 2 \cdot 28 \times 10^{-13}$

$$\text{and} \quad a_H = \frac{\hbar^2}{me^2} = 0 \cdot 532 \times 10^{-8} \text{ cm.}$$

$$f(0) = 1 \cdot 8,$$

we obtain

$$\frac{\triangle \nu_H}{c} \sim 4 \times 10^{-9} \text{ cm.}^{-1}$$

If we observe that the discrepancy actually observed is of the order 10^{-3} cm.$^{-1}$ it will be seen that the correction obtained is negligible.

If, however, the Born-Schrödinger radius be substituted in (17) we have, since this radius is nearly equal to 14 r_0,

$$\frac{\triangle \nu_H}{c} \sim 2 \times 10^{-5} \text{ cm.}^{-1}$$

i.e., only about two per cent. of the discrepancy is explained. We might therefore conclude that the corrections obtained are negligible.

4. *Conclusion.*

I wish to thank Prof. Max Born who suggested this problem to me in the winter of 1935 and also indicated the method of procedure to be adopted.

[15] Bethe, *ibid.*, p. 283.

(4)

SEMI - VECTORS IN BORN'S FIELD THEORY.

Reprinted from "The Proceedings of the Indian Academy of Sciences", Vol. IV, No. 4, Sec. A, 1936.

SEMI-VECTORS IN BORN'S FIELD THEORY.

BY B. S. MADHAVA RAO.

(University of Mysore.)

(From the Department of Physics, Indian Institute of Science, Bangalore.)

Received August 17, 1936.

1. *Introduction.*

IN their first paper[1] on the quantisation of the field equations of the new field theory, Born and Infeld showed that there are four possible standpoints for the choice of the action function needed to develop the field theory. These correspond to the four ways of choosing the primary field vectors (one electric and one magnetic) from among $\vec{E}$, $\vec{D}$, $\vec{H}$, $\vec{B}$, *viz.*, $\vec{E}$, $\vec{B}$ — $\vec{D}$, $\vec{H}$ — $\vec{D}$, $\vec{B}$ — $\vec{E}$, $\vec{H}$. In each case there exists an action function from which the complementary pair can be derived by differentiation. In the first two cases the action functions were respectively the Lagrangian L $(\vec{B}, \vec{E})$ and the Hamiltonian H $(\vec{D}, \vec{H})$ and Born and Infeld have shown in an earlier paper[2] that in both these cases the action functions can be explicitly put, using the antisymmetrical tensors f_{kl} and p_{kl}, in an invariant form and similarly that the relations connecting primary and secondary vectors can be exhibited in a Lorentz-invariant form using tensor notation.

The other two methods of choice which correspond to U $(\vec{D}, \vec{B})$ the energy-density and V $(\vec{E}, \vec{H})$ mean the splitting of the antisymmetrical tensors and hence the abandonment of the four dimensional tensor notation. It is however necessary for the relativistic invariance of the new field theory that the relations

$$\vec{E} = \frac{\partial U}{\partial \vec{D}}, \quad \vec{H} = \frac{\partial U}{\partial \vec{B}}$$

and
$$\vec{D} = -\frac{\partial V}{\partial \vec{E}}, \quad \vec{B} = -\frac{\partial V}{\partial \vec{H}}$$

should be Lorentz-invariant. We show in this paper that the above relations and consequently the field equations derived from U and V are invariant

[1] Born and Infeld, *Proc. Roy. Soc.*, A, 1934, **147**, 522.

[2] *Proc. Roy. Soc.*, A, 1934, **144**, 425.

against Lorentz-transformations although U and V themselves are not Lorentz-invariant. For this purpose we employ the (B)- and (C)-transformations of Einstein and Mayer's semi-vector theory[3] and show that the first set of the above relations are invariant against the (B)- and the second set against the (C)-transformations.

The conditions $(\vec{B} \times \vec{H}) + (\vec{D} \times \vec{E}) = 0$, and $(\vec{D} \times \vec{B}) = (\vec{E} \times \vec{H})$ which are necessary for the relativistic invariance[4] of the field theory are examined for invariance under the above transformations and it is shown that while the first condition is invariant only under the (B)-transformations the second is so under both.

The several invariants are expressed in spinor notation and it is shown therefrom that U and V are not spinor invariants.

2. *U and V not Lorentz-invariant.*

The Lagrangian $L = L(\vec{B}^2 - \vec{E}^2, \vec{B}\cdot\vec{E}) = L(F, G)$ and the Hamiltonian $H = H(\vec{D}^2 - \vec{H}^2, \vec{D}\cdot\vec{H}) = H(P, Q)$
are Lorentz-invariant since F, G, P and Q are tensor invariant but

$$U = L + (\vec{D}\cdot\vec{E})$$

and $$V = L - (\vec{B}\cdot\vec{H})$$

are not Lorentz-invariant since $(\vec{D}\cdot\vec{E})$ and $(\vec{B}\cdot\vec{H})$ are not so being merely space-invariants.

3. *U and V not spinor-invariant.*

We can also prove that U and V are not Lorentz-invariant by showing that they are not spinor invariants. Using the notation of Laporte and Uhlenbeck[5] let

$$G^{kl} = f^{kl} + f^{*kl} \quad \text{(using the Minkowski line element)}$$

be a self-dual antisymmetric tensor. We have

$$\begin{cases} G_{23} = f_{23} - if_{14}, \cdots \\ G_{14} = f_{14} + if_{23}, \cdots \end{cases} \qquad \begin{pmatrix} (f_{23}, f_{31}, f_{12}) \to \vec{B} \\ (f_{14}, f_{24}, f_{34}) \to \vec{E} \end{pmatrix}$$

such that the relations

$$\begin{aligned} G_{23} &= - iG_{14}, \ldots\ldots \\ G_{14} &= iG_{23}, \ldots\ldots \end{aligned}$$ are satisfied.

---

[3] Einstein and Mayer, *Berl. Ber.*, 1932, 522.

[4] See Born and Infeld, *Proc. Roy. Soc.*, A, 1935, **150**, 159.

[5] Laporte and Uhlenbeck, *Phys. Rev.*, 1931, **38** (II), 1380.

Denoting the "*Kenn-zahlen*" of G^{kl} by k_1, k_2, k_3 we can write

$$G^{kl} = \begin{pmatrix} 0 & k_3 & -k_2 & -ik_1 \\ -k_3 & 0 & k_1 & -ik_2 \\ k_2 & -k_1 & 0 & -ik_3 \\ ik_1 & ik_2 & ik_3 & 0 \end{pmatrix}$$

from which we easily deduce $G_{23} = -k_1 = f_{23} - if_{14}$; $G_{31} = -k_2 =$
$$f_{31} - if_{24}\ ;\ G_{12} = k_3 = f_{12} - if_{34}.$$

We can associate with G^{kl} the two conjugate symmetric spinors g_{rs} and $g^{\dot{r}\dot{s}}$ whose elements are given by

$$g_{\dot{1}\dot{1}} = 2\,(k_2 + ik_1)\ ;\ g_{\dot{2}\dot{2}} = 2\,(k_2 - ik_1)\ ;\ g_{\dot{1}\dot{2}} = g_{\dot{2}\dot{1}} = -\,2ik_3$$

$$g_{11} = 2\,(\bar{k}_2 - i\bar{k}_1)\ ;\ g_{22} = 2\,(\bar{k}_2 + i\bar{k}_1)\ ;\ g_{12} = g_{21} = 2i\bar{k}_3.$$

We can form the two spinor-invariants

$$g_{rs}\, g^{rs} \text{ and } g_{\dot{r}\dot{s}}\, g^{\dot{r}\dot{s}}.$$

$$g_{rs}\, g^{rs} = g_{11}\, g^{11} + g_{12}\, g^{12} + g_{21}\, g^{21} + g_{22}\, g^{22} = 2\,(g_{11}\, g_{22} - g_{12}{}^2)$$

$$\begin{aligned} \therefore\quad \tfrac{1}{8}\, g_{rs}\, g^{rs} &= (\bar{k}_2 - i\bar{k}_1)(\bar{k}_2 + i\bar{k}_1) - (i\bar{k}_3)^2 = \bar{k}_1{}^2 + \bar{k}_2{}^2 + \bar{k}_3{}^2 \\ &= (f_{23} + if_{14})^2 + (f_{31} + if_{24})^2 + (f_{12} + if_{34})^2 \\ &= F + 2iG. \end{aligned}$$

In exactly the same way, we have

$$\tfrac{1}{8}\, g_{\dot{r}\dot{s}}\, g^{\dot{r}\dot{s}} = k_1{}^2 + k_2{}^2 + k_3{}^2 = F - 2iG$$

which proves the invariance of F and G.

Next consider, $H^{kl} = p^{kl} + p^{*kl}$ $\quad \left(\begin{array}{l} p_{23}, p_{31}, p_{12} \rightarrow \vec{H} \\ p_{14}, p_{24}, p_{34} \rightarrow \vec{D} \end{array} \right)$

and associate the spinors h_{rs} and $h^{\dot{r}\dot{s}}$ defined by l_1, l_2, l_3. The invariants $h_{rs}\, h^{rs}$ and $h_{\dot{r}\dot{s}}\, h^{\dot{r}\dot{s}}$ give exactly as above the two spinor-invariants

$$\tfrac{1}{8}\, h_{rs}\, h^{rs} = P + 2iQ$$

$$\tfrac{1}{8}\, h_{\dot{r}\dot{s}}\, h^{\dot{r}\dot{s}} = P - 2iQ$$

which proves the invariance of P and Q.

We can also consider the mixed invariants

$$g_{rs}\, h^{rs} \text{ and } g_{\dot{r}\dot{s}}\, h^{\dot{r}\dot{s}}.$$

We have

$$g_{rs}\, h^{rs} = g_{11}\, h^{11} + g_{12}\, h^{12} + g_{21}\, h^{21} + g_{22}\, h^{22} = g_{11}\, h_{22} + g_{22}\, h_{11} - 2g_{12}\, h_{12}$$

$$i.e., \ \tfrac{1}{4}\, g_{rs}\, h^{rs} = (\bar{k}_2 - i\bar{k}_1)\,(\bar{l}_2 + i\bar{l}_1) + (\bar{k}_2 + i\bar{k}_1)\,(\bar{l}_2 - i\bar{l}_1) - 2(i\bar{k}_3)\,(i\bar{l}_3)$$

$$\text{and } \tfrac{1}{8}\, g_{rs}\, h^{rs} = \bar{k}_1\, \bar{l}_1 + \bar{k}_2\, \bar{l}_2 + \bar{k}_3\, \bar{l}_3 = (f_{23} - if_{14})\,(p_{23} - ip_{14}) + \cdots$$

$$= (\vec{B}\vec{H} - \vec{E}\vec{D}) - i\,(\vec{B}\vec{D} + \vec{E}\vec{H}).$$

In exactly the same way, we find

$$\tfrac{1}{8}\, g_{\dot r \dot s}\, h^{\dot r \dot s} = k_1\, l_1 + k_2\, l_2 + k_3\, l_3$$
$$= (\vec{B}\vec{H} - \vec{E}\vec{D}) + i\,(\vec{B}\vec{D} + \vec{E}\vec{H})$$

which proves the invariance of $(\vec{B}\vec{H} - \vec{E}\vec{D})$; $(\vec{B}\vec{D} + \vec{E}\vec{H})$.

We have so far proved that $(\vec{B}^2 - \vec{E}^2)$, $(\vec{B}\vec{E})$; $(\vec{H}^2 - \vec{D}^2)$, $(\vec{H}\vec{D})$; $(\vec{B}\vec{H} - \vec{E}\vec{D})$ and $(\vec{B}\vec{D} + \vec{E}\vec{H})$ are spinor-invariants but these are also tensor invariants as evident from the following forms

$$\left.\begin{aligned}
&\vec{B}^2 - \vec{E}^2 = \tfrac{1}{2} f_{kl} f^{kl} ;\ (\vec{B}\vec{E}) = 1/4\, f_{sk} f^{*sk} \\
&\vec{D}^2 - \vec{H}^2 = \tfrac{1}{2} p^{*kl} p_{kl}^{*} = \tfrac{1}{2} p_{kl} p^{kl} ;\ (\vec{H}\vec{D}) = 1/4\, p^{sk} p_{sk}^{*} \\
&\vec{B}\vec{H} - \vec{E}\vec{D} = \tfrac{1}{2} p^{kl} f_{kl}\ \ \vec{B}\vec{D} + \vec{E}\vec{H} = \tfrac{1}{2} p^{*kl} f_{kl}
\end{aligned}\right\}$$

In terms of k_1, k_2, k_3 and l_1, l_2, l_3 these invariants are derived from the forms

$$k_1^2 + k_2^2 + k_3^2 ;\ \bar{k}_1^2 + \bar{k}_2^2 + \bar{k}_3^2 ;\ l_1^2 + l_2^2 + l_3^2 ;\ \bar{l}_1^2 + \bar{l}_2^2 + \bar{l}_3^2,$$
$$k_1\, l_1 + k_2\, l_2 + k_3\, l_3 \text{ and } \bar{k}_1\, \bar{l}_1 + \bar{k}_2\, \bar{l}_2 + \bar{k}_3\, \bar{l}_3.$$

We can now form the four expressions

$$k_1 \bar{k}_1 + k_2 \bar{k}_2 + k_3 \bar{k}_3 ;\ l_1 \bar{l}_1 + l_2 \bar{l}_2 + l_3 \bar{l}_3$$
$$k_1\, l_1 + \bar{k}_2\, l_2 + \bar{k}_3\, l_3 ;\ k_1\, \bar{l}_1 + k_2\, \bar{l}_2 + k_3\, l_3$$

which reduce to

$$\vec{B}^2 + \vec{E}^2 ;\ \vec{H}^2 + \vec{D}^2 ;\ (\vec{B}\vec{H} + \vec{E}\vec{D}) + i(\vec{E}\vec{H} - \vec{B}\vec{D}) ;\ (\vec{B}\vec{H} + \vec{E}\vec{D}) - i(\vec{E}\vec{H} - \vec{B}\vec{D}).$$

These are not however spinor-invariants since they correspond to the forms $g_{rs}\, g^{\dot r \dot s}$; $h_{rs}\, h^{\dot r \dot s}$; $g_{rs}\, h^{\dot r \dot s}$; $g^{\dot r \dot s}\, h^{rs}$ and contraction of dotted and undotted indices does not give such an invariant. Thus $\vec{B}\vec{H} + \vec{E}\vec{D}$ is not a spinor-invariant while $\vec{B}\vec{H} - \vec{E}\vec{D}$ is so. Hence $(\vec{D}\vec{E})$ and $(\vec{B}\vec{H})$ are not spinor, *i.e.*, Lorentz-invariants and the same is consequently true of U and V.

4. *Semi-Vectors.*

We will give here those parts of the Einstein-Mayer theory of semi-vectors which will be used in the present investigation. Since Einstein-Mayer use the Minkowski line-element we shall transfer the results to the case of the line element used in the field theory.

According to the line-element used in the Born-Infeld theory we have the relation

$$(h_{ik}^{*})^{*} = -\, h_{ik}$$

so that, if $h_{ik}^{*} = a h_{ik}$.

61

We easily derive $a^2 = -1$, *i.e.*, $a = \pm i$; and, hence the self-dual and anti-dual antisymmetric tensors are defined by

$$u_{ik}^{*} = iu_{ik} \text{ and } v_{ik}^{*} = -iv_{ik}$$

and using the rule of pulling the indices up and down

$$\left.\begin{array}{l} u_{23} = -iu_{14};\ u_{14} \quad = iu_{23}, \text{ etc.} \\ v_{23} = \quad iv_{14};\ v_{14} -= iv_{23}, \text{ etc.} \end{array}\right\}$$

just as in the Einstein-Mayer theory. The special antisymmetric u's and v's form a linear space and because of this linearity any u_{ik} (or v_{ik}) can be expressed in terms of three suitably chosen u_{ik}'s (or v_{ik}'s). Let us take $\underset{1}{u}_{ik}$ defined such that all except $\underset{1}{u}_{23}$ is equal to zero, but $\underset{1}{u}_{23} = -iu_{14} = 1$ with $\underset{2}{u}_{ik}$ and $\underset{3}{u}_{ik}$ similarly defined. We can then form any u_{ik} by

$$\left.\begin{array}{ll} & u_{ik} = \alpha_1 \underset{1}{u}_{ik} + \alpha_2 \underset{2}{u}_{ik} + \alpha_3 \underset{3}{u}_{ik} \\ \text{and similarly} & v_{ik} = \beta_1 \underset{1}{v}_{ik} + \beta_2 \underset{2}{v}_{ik} + \beta_3 \underset{3}{v}_{ik} \end{array}\right\}$$

where $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ are quite arbitrary. We can thus write

$$\left.\begin{array}{l} u_{23} = \alpha_1,\ u_{31} = \alpha_2,\ u_{12} = \alpha_3,\ u_{14} = i\alpha_1, \cdots \\ v_{23} = \beta_1,\ v_{31} = \beta_2,\ v_{12} = \beta_3,\ v_{14} = -i\beta_1, \cdots \end{array}\right\} \quad \ldots \quad \ldots \quad (1)$$

Let (a_{ik}) be the group (D) of rotations constituting the Lorentz-group and let us write

$$a_{ik} = b_{ip}\, c_k^{p}$$

where (b_{ik}), (c_{ik}) are two sub-groups of the Lorentz-group and also isomorphous with (a_{ik}). The condition for the existence of such a factorisation is that the b- and c-rotations be "*vertauschbar*". When this condition is satisfied and when c_{ik} is taken as the infinitesimal rotation

$$c_{ik} = g_{ik} + v_{ik}$$

we can determine b_{ik} from the necessary consequence that (b_{ik}) is "*austachbar*" with (v_{ik}). We can find for b_{ik} the form

$$b_{ik} = bg_{ik} + u_{ik}$$

where $b = b_{11} = b_{22} = b_{33} = -b_{44}$. In general the (b_{ik}) so determined will not be a "*Drehung*" and will be one if

$$b^2 + \tfrac{1}{4}\, u_{ik}\, u^{ki} = 1.$$

The group of rotations in (b_{ik}), *i.e.*, those satisfying the above condition is denoted by (B) and is the greatest common measure of (a_{ik}) and (b_{ik}). We can similarly define the sub-group of rotations (C).

An infinitesimal element of the group (b_{ik}) is given by $g_{ik}\,(1 + \delta b) + \delta u_{ik}$ $(b_{ik} = bg_{ik} + u_{ik})$ and the condition, for this being a rotation, *viz.*,

$b^2 + \frac{1}{4} u_{ik} u^{ki} = 1$, becomes

$$(1 + \delta b)^2 + \tfrac{1}{4} \delta u_{ik} \delta u^{ik} = 1 \text{ or } \delta b = 0.$$

Hence an infinitesimal element of (B) is $g_{ik} + \delta u_{ik}$; similarly an infinitesimal element of the group (C) is given by

$$g_{ik} + \delta v_{ik}, \text{ i.e., by}$$

$$\left.\begin{aligned} b_{ik} &= g_{ik} + \delta u_{ik} \\ c_{ik} &= g_{ik} + \delta v_{ik} \end{aligned}\right\} \qquad (2)$$

5. Equations of transformation.

The equations connecting field quantities[6] can be written in the form,

$$\left.\begin{aligned} \vec{E} &= \frac{\vec{D}(1 + \vec{B}^2) - \vec{B}R}{\sqrt{(1 + \vec{D}^2)(1 + \vec{B}^2) - R^2}} \\ \vec{H} &= \frac{\vec{B}(1 + \vec{D}^2) - \vec{D}R}{\sqrt{(1 + \vec{D}^2)(1 + \vec{B}^2) - R^2}} \end{aligned}\right\} \qquad (3)$$

where $U = \sqrt{(1 + \vec{D}^2)(1 + \vec{B}^2) - R^2}$, and $R = (\vec{D} \cdot \vec{B})$.

We will show that the equations (3) are invariant with respect to transformations of the group (B) and thus establish their Lorentz-invariance. Since group (B) consists of rotations, it is sufficient to consider only infinitesimal elements of the group.

Let us denote by

$$x^1, x^2, x^3, x^4 \longrightarrow x, y, z, t$$
$$x^{1\prime}, x^{2\prime}, x^{3\prime}, x^{4\prime} \longrightarrow x', y', z', t'$$

the co-ordinates of world-points in systems Σ and Σ' connected by infinitesimal transformations defined by (2).

In consonance with (2) we can write such a transformation as

$$x'^{\mu} = g_{\nu}{}^{\mu} x^{\nu} + \delta u_{\nu}{}^{\cdot\mu} x^{\nu} \qquad (4)$$

such that $g_{\nu}{}^{\mu} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ if } \mu \neq \nu \end{cases}$

and the dot in $u_{\nu}{}^{\cdot\mu}$ denotes the order of the indices. The tensor components f_{ik} and p_{ik} are derived from the transformation rules of tensors. Thus,

$$\left.\begin{aligned} f'_{14} &= (g_1{}^{\nu} + \delta u_1{}^{\nu})(g_4{}^{\rho} + \delta u_4{}^{\rho}) f_{\nu\rho} = f_{14} + \delta u_1{}^{\nu} f_{\nu 4} + \delta u_4{}^{\nu} f_{1\nu} \\ f'_{23} &= (g_2{}^{\nu} + \delta u_2{}^{\nu})(g_3{}^{\rho} + \delta u_3{}^{\rho}) f_{\nu\rho} = f_{23} + \delta u_2{}^{\nu} f_{\nu 3} + \delta u_3{}^{\nu} f_{2\nu} \end{aligned}\right\}$$

where, in consequence of the infinitesimal nature of the transformation, we

[6] Born and Infeld, Ref. (1), p. 525, formula (2.7A).

neglect the coefficient of the second power of δ. Writing these in the expanded form, we get

$$\left.\begin{aligned} f'_{14} &= f_{14} + \delta\alpha_2\,(f_{34} + if_{12}) - \delta\alpha_3\,(f_{24} + if_{31}) \\ f'_{23} &= f_{23} + \delta\alpha_2\,(f_{12} - if_{34}) - \delta\alpha_3\,(f_{31} - if_{24}) \end{aligned}\right\} \quad \ldots \quad (5)$$

making using of (1). Introducing a space-vector $\vec{\alpha}$ whose components are $(\alpha_1, \alpha_2, \alpha_3)$ these equations can be written as

$$\left.\begin{aligned} \vec{E}' &= \vec{E} + i\delta\,(\vec{\alpha} \times \overrightarrow{B - iE})\,;\ \vec{B}' = \vec{B} + \delta\,(\vec{\alpha} \times \overrightarrow{B - iE}) \\ \vec{D}' &= \vec{D} + i\delta\,(\vec{\alpha} \times \overrightarrow{H - iD})\,;\ \vec{H}' = \vec{H} + \delta\,(\vec{\alpha} \times \overrightarrow{H - iD}) \end{aligned}\right\} \quad \ldots \quad (6)$$

the second set in (6) being obtained by the transformation of the components of the tensor p_{kl}. Dealing with the (C)-Lorentz-transformation in a similar way we have the equations of transformation

$$\left.\begin{aligned} f'_{14} &= f_{14} - i\delta\beta_2\,(f_{12} + if_{34}) + i\delta\beta_3\,(f_{31} + if_{24}) \\ f'_{23} &= f_{23} + \delta\beta_2\,(f_{12} + if_{34}) - \delta\beta_3\,(f_{31} + if_{24}) \end{aligned}\right\} \quad \ldots \quad (5')$$

or taking $\vec{\beta} \rightarrow (\beta_1, \beta_2, \beta_3)$,

$$\left.\begin{aligned} \vec{E}' &= \vec{E} - i\delta\,(\vec{\beta} \times \overrightarrow{B + iE})\,;\ \vec{B}' = \vec{B} + \delta\,(\vec{\beta} \times \overrightarrow{B + iE}) \\ \vec{D}' &= \vec{D} - i\delta\,(\vec{\beta} \times \overrightarrow{H + iD})\,;\ \vec{H}' = \vec{H} + \delta\,(\vec{\beta} \times \overrightarrow{H + iD}) \end{aligned}\right\} \quad \ldots \quad (6')$$

6. *Lorentz-invariance of the conditions.*

$$(\vec{E} \times \vec{H}) - (\vec{D} \times \vec{B}) = 0\,;\ (\vec{E} \times \vec{D}) + (\vec{H} \times \vec{B}) = 0.$$

Before proceeding to prove the invariance of the equations (3) we will consider the transformation of the above two conditions

$$(\vec{E} \times \vec{H}) - (\vec{D} \times \vec{B}) = 0 \quad \ldots \quad (7)$$

and $$(\vec{E} \times \vec{D}) + (\vec{H} \times \vec{B}) = 0 \quad \ldots \quad (8)$$

under (6) and (6′). The conditions (7) and (8) are intimately connected with the question of the Lorentz-invariance of the field equations derived from U. In fact, it is known [*cf.*, Born-Infeld, Reference (4)] that the former is a necessary and sufficient condition while the latter is a necessary one only. We will now show that (7) remains invariant under both the (B)- and (C)-transformations while (8) does so only under the (B)-transformation. Under transformation (6),

$$\begin{aligned} (\vec{D}' \times \vec{B}') - (\vec{E}' \times \vec{H}') = {} & \left\{\vec{D} + i\delta\,(\vec{\alpha} \times \overrightarrow{H - iD})\right\} \times \left\{\vec{B} + \delta\,(\vec{\alpha} \times \overrightarrow{B - iE})\right\} \\ & - \left\{\vec{E} + i\delta\,(\vec{\alpha} \times \overrightarrow{B - iE})\right\} \times \left\{\vec{H} + \delta\,(\vec{\alpha} \times \overrightarrow{H - iD})\right\} \end{aligned}$$

$$= \delta \vec{D} \times (\vec{\alpha} \times \overrightarrow{B - iE}) - i\delta \vec{B} \times (\vec{\alpha} \times \overrightarrow{H - iD}) - \delta\vec{E}$$
$$\times (\vec{\alpha} \times \overrightarrow{H - iD}) + i\delta \vec{H} \times (\vec{\alpha} \times \overrightarrow{B - iE})$$
$$[(\text{since } (\vec{D} \times \vec{B}) - (\vec{E} \times \vec{H}) = 0 \text{ from } (8)]$$
$$= i\delta \left\{\vec{E}(\vec{D}\cdot\vec{\alpha}) - \vec{D}(\vec{E}\cdot\vec{\alpha})\right\} + i\delta\left\{\vec{H}(\vec{B}\cdot\vec{\alpha}) - \vec{B}(\vec{H}\cdot\vec{\alpha})\right\}$$
$$+ \delta\left\{\vec{H}(\vec{E}\cdot\vec{\alpha}) - \vec{E}(\vec{H}\cdot\vec{\alpha})\right\} + \delta\left\{\vec{D}(\vec{B}\cdot\vec{\alpha}) - \vec{B}(\vec{D}\cdot\vec{\alpha})\right\}$$
$$[\text{since all terms in } \vec{\alpha} \text{ cancel out}]$$
$$= i\delta\,\vec{\alpha}\times(\vec{E}\times\vec{D}) + i\delta\,\vec{\alpha}\times(\vec{H}\times\vec{B}) + \delta\vec{\alpha}\times(\vec{H}\times\vec{E}) + \delta\vec{\alpha}\times(\vec{D}\times\vec{B})$$
$$= 0, \text{ using (7) and (8).}$$

Under transformation (6′),

$$(\vec{D}'\times\vec{B}') - (\vec{E}'\times\vec{H}')$$
$$= i\delta\vec{B}\times(\vec{\beta}\times\overrightarrow{H + iD}) - i\delta\vec{H}\times(\vec{\beta}\times\overrightarrow{B + iE})$$
$$+ \delta\vec{D}\times(\vec{\beta}\times\overrightarrow{B + iE}) - \delta\vec{E}\times(\vec{\beta}\times\overrightarrow{H + iD})$$
$$= i\delta\left\{\vec{B}(\vec{H}\cdot\vec{\beta}) - \vec{H}(\vec{B}\cdot\vec{\beta})\right\} + i\delta\left\{\vec{D}(\vec{E}\cdot\vec{\beta}) - \vec{E}(\vec{D}\cdot\vec{\beta})\right\}$$
$$+ \delta\left\{\vec{H}(\vec{E}\cdot\vec{\beta}) - \vec{E}(\vec{H}\cdot\vec{\beta})\right\} + \delta\left\{\vec{D}(\vec{B}\cdot\vec{\beta}) - \vec{B}(\vec{D}\cdot\vec{\beta})\right\}$$
$$= i\delta\vec{\beta}\times\left\{(\vec{B}\times\vec{H}) + (\vec{D}\times\vec{E})\right\} + \delta\vec{\beta}\times\left\{(\vec{H}\times\vec{E}) + (\vec{D}\times\vec{B})\right\}$$
$$= 0, \text{ using (7) and (8).}$$

Under transformation (6),

$$(\vec{E}'\times\vec{D}') + (\vec{H}'\times\vec{B}') = i\delta\vec{\alpha}\times(\vec{E}\times\vec{H}) + i\delta\vec{\alpha}\times(\vec{B}\times\vec{D}) + \delta\vec{\alpha}$$
$$\times(\vec{H}\times\vec{B}) + \delta\vec{\alpha}\times(\vec{E}\times\vec{D})$$
$$= 0.$$

Finally, under transformation (6′),

$$(\vec{E}'\times\vec{D}') + (\vec{H}'\times\vec{B}') = 2i\delta\vec{\beta}\,(\vec{D}\vec{B} - \vec{E}\vec{H}) - i\delta\left\{\vec{B}(\vec{D}\cdot\vec{\beta}) + \vec{D}(\vec{B}\cdot\vec{\beta})\right\}$$
$$+ i\delta\left\{\vec{H}(\vec{E}\cdot\vec{\beta}) + \vec{E}(\vec{H}\cdot\vec{\beta})\right\} + 2\delta\vec{\beta}\times(\vec{E}\times\vec{D})$$

and the coefficients multiplying $\vec{\beta}$ scalarly and vectorially do not vanish. Remembering that $\vec{\beta}$ is an arbitrary vector with the " Kennzahlen " $\beta_1, \beta_2, \beta_3$ belonging to an arbitrary anti-dual antisymmetric v_{ik}, this shows that condition (8) is not invariant under the (C)-transformations.

7. *Lorentz-invariance of the field equations.*

Consider now the right-hand side of the first of the equations (3) and let us determine how it is transformed under (6).

$$\vec{D}'^2 = \vec{D}^2 + 2i\delta\vec{\alpha}\,(\overrightarrow{\vec{H} - i\vec{D}} \times \vec{D}) = \vec{D}^2 + 2i\delta\vec{\alpha}\,(\vec{H} \times \vec{D})$$

$$\vec{B}'^2 = \vec{B}^2 + 2i\delta\vec{\alpha}\,(\overrightarrow{\vec{B} - i\vec{E}} \times \vec{B}) = \vec{B}^2 + 2i\delta\vec{\alpha}\,(\vec{B} \times \vec{E})$$

$$(\vec{B}' \cdot \vec{D}') = (\vec{B} \cdot \vec{D}) + \delta\vec{\alpha}\,(\overrightarrow{\vec{B} - i\vec{E}} \times \vec{D}) + i\delta\vec{\alpha}\,(\overrightarrow{\vec{H} - i\vec{D}} \times \vec{B})$$

$$= (\vec{B} \cdot \vec{D}) - i\delta\vec{\alpha}\,(\vec{E} \times \vec{D}) + i\delta\vec{\alpha}\,(\vec{H} \times \vec{B})$$

i.e., $$R' = R + 2i\delta\vec{\alpha}\,(\vec{H} \times \vec{B}) = R - 2i\delta\vec{\alpha}\,(\vec{E} \times \vec{D}) \text{ using (8)}$$

$$\therefore \quad R'^2 = R^2 + 4i\delta R\vec{\alpha}\,(\vec{H} \times \vec{B})$$

and $(1 + \vec{D}'^2)(1 + \vec{B}'^2) - R'^2 = \Omega + 2i\delta\,\Omega_2$ .. .. .. (9)

where $\Omega = (1 + \vec{D}^2)(1 + \vec{B}^2) - R^2$

and $\Omega_2 = (1 + \vec{B}^2)\,\vec{\alpha} \cdot (\vec{H} \times \vec{D}) + (1 + \vec{D}^2)\,\vec{\alpha} \cdot (\vec{B} \times \vec{E}) - 2R\vec{\alpha}\,(\vec{H} \times \vec{B})$.

Similarly for the numerator,

$$\vec{D}'(1 + \vec{B}'^2) - \vec{B}'R' = \left\{\vec{D}(1 + \vec{B}^2) - \vec{B}R\right\} + \delta\vec{\Omega}_1 \quad \ldots \quad \ldots (10)$$

where $\vec{\Omega}_1 = i(1 + \vec{B}^2)(\vec{\alpha} \times \overrightarrow{\vec{H} - i\vec{D}}) + 2i\vec{D}\left\{\vec{\alpha} \cdot (\vec{B} \times \vec{E})\right\}$

$$- 2i\vec{B}\left\{\vec{\alpha} \cdot (\vec{H} \times \vec{B})\right\} - R(\vec{\alpha} \times \overrightarrow{\vec{B} - i\vec{E}}).$$

Hence the expression

$$\frac{\vec{D}'(1+\vec{B}'^2) - \vec{B}'R'}{\sqrt{(1+\vec{D}'^2)(1+\vec{B}'^2) - R'^2}} = \frac{\left\{\vec{D}(1+\vec{B}^2) - \vec{B}R\right\} + \delta\vec{\Omega}_1}{\sqrt{\Omega + 2i\delta\Omega_2}}, \text{ from (9) and (10)}$$

$$= \frac{\left\{\vec{D}(1+\vec{B}^2) - \vec{B}R\right\} + \delta\vec{\Omega}_1}{\sqrt{\Omega}}\left\{1 + \frac{2i\delta\Omega_2}{\Omega}\right\}^{-\frac{1}{2}}$$

$$= \frac{\left[\left\{\vec{D}(1+\vec{B}^2) - \vec{B}R\right\} + \delta\vec{\Omega}_1\right]\left[1 - \frac{i\delta\Omega_2}{\Omega}\right]}{\sqrt{\Omega}}$$

$$= \frac{\vec{D}(1+\vec{B}^2) - \vec{B}R}{\sqrt{\Omega}} + \frac{\delta\left\{\vec{\Omega}_1 - i\Omega_2\vec{E}/\sqrt{\Omega}\right\}}{\sqrt{\Omega}}, \text{ using (3)}$$

$$= \vec{E} + \frac{\delta\vec{\Omega}_1}{\sqrt{\Omega}} - \frac{i\delta\Omega_2\vec{E}}{\Omega} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (11)$$

Now, $\Omega_2 = \dfrac{2\,(1+\vec{B}^2)\,(1+\vec{D}^2)}{\sqrt{\Omega}}\,\vec{\alpha}\,(\vec{B}\times\vec{D}) - 2R\vec{\alpha}\left\{\dfrac{\vec{B}\,(1+\vec{D}^2)-\vec{D}R}{\sqrt{\Omega}}\times\vec{B}\right\}$

[substituting for $\vec{H}$ and $\vec{E}$ from (3)]

$$= 2\sqrt{\Omega}\;\vec{\alpha}\,(\vec{B}\times\vec{D}) \quad \ldots \quad (12)$$

Also $\Omega_1 = \left\{(1+\vec{B}^2)\,(\vec{\alpha}\times\vec{D}) - R\,(\vec{\alpha}\times\vec{B})\right\} + i\,(1+\vec{B}^2)\,(\vec{\alpha}\times\vec{H})$

$$+ iR\,(\vec{\alpha}\times\vec{E}) + 2\,i\vec{D}\left\{\vec{\alpha}\,(\vec{B}\times\vec{E})\right\} - 2i\vec{B}\,\{\vec{\alpha}\,(\vec{H}\times\vec{B})\}$$

$$= \sqrt{\Omega}\,(\vec{\alpha}\times\vec{E}) + i\,(1+\vec{B}^2)\left\{\vec{\alpha}\times\frac{\vec{B}\,(1+\vec{D}^2)-\vec{D}R}{\sqrt{\Omega}}\right\}$$

$$+ iR\left\{\vec{\alpha}\times\frac{\vec{D}\,(1+\vec{B}^2)-\vec{B}R}{\sqrt{\Omega}}\right\}$$

$$+ 2i\vec{D}\left\{\vec{\alpha}\,(\vec{B}\times\vec{E})\right\} + 2i\vec{B}\left\{\vec{\alpha}\,(\vec{E}\times\vec{D})\right\}$$

$$= \sqrt{\Omega}\,(\vec{\alpha}\times\vec{E}) + i\sqrt{\Omega}\,(\vec{\alpha}\times\vec{B}) + 2i\vec{D}\,(\vec{\alpha}\vec{B}\vec{E}) + 2i\vec{B}\,(\vec{\alpha}\vec{E}\vec{D}) \quad \ldots (13)$$

(the last two terms containing scalar triple products within brackets).

Putting (12) and (13) in (11) we get

$$\text{L. H. S. of (11)} = \vec{E} + \delta\,(\vec{\alpha}\times\vec{E}) + i\delta\,(\vec{\alpha}\times\vec{B}) + \frac{2i\delta}{\sqrt{\Omega}}\left\{\vec{D}\,(\vec{\alpha}\,\vec{B}\,\vec{E}) + \vec{B}\,(\vec{\alpha}\,\vec{E}\,\vec{D}) - \vec{E}\,(\vec{\alpha}\,\vec{B}\,\vec{D})\right\}.$$

Now for any four vectors a, b, c, d we can deduce the identity

$$\vec{a}\,(\vec{b}\,\vec{c}\,\vec{d}) - \vec{b}\,(\vec{a}\,\vec{c}\,\vec{d}) + \vec{c}\,(\vec{a}\,\vec{b}\,\vec{d}) - \vec{d}\,(\vec{a}\,\vec{b}\,\vec{c}) = 0$$

and using this identity the terms within { } on the right hand side of the above expression reduce to

$\vec{\alpha}\,(\vec{D}\,\vec{B}\,\vec{E})$ which is equal to

$$= \vec{\alpha}\,(\vec{B}\,\vec{E}\,\vec{D}) = \vec{\alpha}\left\{\vec{B}\cdot(\vec{E}\times\vec{D})\right\} = \vec{\alpha}\left\{\vec{B}\cdot(\vec{B}\times\vec{H})\right\}$$

$$= \vec{\alpha}\left\{\vec{H}\cdot(\vec{B}\times\vec{B})\right\} = 0.$$

Hence,

$$\frac{\vec{D}'\,(1+\vec{B}'^2) - \vec{B}'R'}{\sqrt{(1+\vec{D}'^2)\,(1+\vec{B}'^2) - R'^2}} = \vec{E} + \delta\,(\vec{\alpha}\times\vec{E}) + i\delta\,(\vec{\alpha}\times\vec{B})$$

$$= \vec{E} + i\delta\,(\vec{\alpha}\times\vec{B} - i\vec{E})$$

$$= \vec{E}' \quad \ldots \quad (14)$$

[using (6)]

Hence we have proved the Lorentz-invariance of the first of the equations (3).

We can similarly prove the invariance of the second of the equations (3), considering $\dfrac{\vec{B}'(1+\vec{D}'^2)-\vec{D}'R'}{\sqrt{(1+\vec{D}'^2)(1+\vec{B}'^2)-R'^2}}$, the denominator is given by exactly the same expression (9). As regards the numerator we can write it in the form

$$\{\vec{B}(1+\vec{D}^2)-\vec{D}R\}+\delta\vec{\Omega}_3 \quad \ldots \quad (10')$$

where $\vec{\Omega}_3=(1+\vec{D}^2)(\vec{\alpha}\times\overrightarrow{B-iE})-iR(\vec{\alpha}\times\overrightarrow{H-iD})+2i\vec{B}(\vec{\alpha}\vec{H}\vec{D})$
$-2i\vec{D}(\vec{\alpha}\vec{H}\vec{B})$,

and in place of (11) we have the expression

$$\vec{H}+\frac{\delta\vec{\Omega}_3}{\sqrt{\Omega}}-\frac{i\delta\Omega_2\vec{H}}{\Omega} \quad \ldots \quad (11')$$

where $\Omega_2=2\sqrt{\Omega}\,(\vec{\alpha}\vec{B}\vec{D})$ .. .. (12′)

and $\vec{\Omega}_3=\{\vec{\alpha}\times[(1+\vec{D}^2)\vec{B}-\vec{D}R]\}-i(1+\vec{D}^2)(\vec{\alpha}\times\vec{E})-iR(\vec{\alpha}\times\vec{H})$
$+2i\vec{B}(\vec{\alpha}\vec{H}\vec{B})-2i\vec{D}(\vec{\alpha}\vec{H}\vec{B})$

$$=\sqrt{\Omega}(\vec{\alpha}\times\vec{H})-i(1+\vec{D}^2)\left\{\vec{\alpha}\times\frac{\vec{D}(1+\vec{B}^2)-\vec{B}R}{\sqrt{\Omega}}\right\}$$

$$-iR\left\{\alpha\times\frac{\vec{B}(1+\vec{D}^2)-\vec{D}R}{\sqrt{\Omega}}\right\}+2i\vec{B}(\vec{\alpha}\vec{H}\vec{B})-2i\vec{D}(\vec{\alpha}\vec{H}\vec{B}).$$

$$=\sqrt{\Omega}(\vec{\alpha}\times\vec{H})-i\sqrt{\Omega}(\vec{\alpha}\times\vec{D})+2i\vec{B}(\vec{\alpha}\vec{H}\vec{B})-2i\vec{D}(\vec{\alpha}\vec{H}\vec{B}) \quad \ldots \quad (13')$$

∴ (11′) reduces to

$$\vec{H}+\delta(\vec{\alpha}\times\vec{H})-i\delta(\vec{\alpha}\times\vec{D})+\frac{2i\delta}{\sqrt{\Omega}}\left\{\vec{B}(\vec{\alpha}\vec{H}\vec{D})-\vec{D}(\vec{\alpha}\vec{H}\vec{B})-\vec{H}(\vec{\alpha}\vec{B}\vec{D})\right\}$$

From the identity of triple scalar products mentioned above the last term within the { } reduces to

$$\begin{aligned}&\vec{\alpha}(\vec{B}\vec{H}\vec{D})\\&=\vec{\alpha}(\vec{D}\vec{B}\vec{H})=\vec{\alpha}(\vec{D}\vec{E}\vec{D}),\text{ using (8)}\\&=0.\end{aligned}$$

Hence we have shown that

$$\begin{aligned}\frac{\vec{B}'(1+\vec{D}'^2)-\vec{D}'R'}{\sqrt{(1+\vec{D}'^2)(1+\vec{B}'^2)-R'^2}}&=\vec{H}+\delta(\vec{\alpha}\times\vec{H})-i\delta(\vec{\alpha}\times\vec{D})\\&=\vec{H}+\delta(\vec{\alpha}\times\overrightarrow{H-iD})\\&=\vec{H}',\text{ using (16)}\end{aligned}$$

which proves the Lorentz-invariance of the second of the equations (3).

8. Equations when $\vec{E}$ and $\vec{H}$ are independent variables.[7]

Starting from the Lagrangian

$$L = \sqrt{1 + (\vec{B}^2 - \vec{E}^2) - G^2} - 1 \quad .. \quad .. \quad .. \quad (a)$$

where $G = (\vec{E} \cdot \vec{B}) \quad .. \quad .. \quad .. \quad .. \quad .. \quad (b)$

we want to express the function

$$V = L - (\vec{B}\vec{H}) \quad .. \quad .. \quad .. \quad .. \quad (c)$$

as function of $\vec{E}$ and $\vec{H}$.

From the equations derived from L, we have

$$\vec{H} = \frac{\partial L}{\partial \vec{B}} = \frac{\vec{B} - G\vec{E}}{\sqrt{1 + \vec{B}^2 - \vec{E}^2 - G^2}} \quad .. \quad .. \quad .. \quad (d)$$

therefore

$$(\vec{B} \cdot \vec{H}) = \frac{\vec{B}^2 - G^2}{\sqrt{1 + \vec{B}^2 - \vec{E}^2 - G^2}}$$

and from (c) using (a)

$$V = \sqrt{1 + \vec{B}^2 - \vec{E}^2 - G^2} - \frac{\vec{B}^2 - G^2}{\sqrt{1 + \vec{B}^2 - \vec{E}^2 - G^2}} - 1$$

$$V = \frac{1 - \vec{E}^2}{\sqrt{1 + \vec{B}^2 - \vec{E}^2 - G^2}} - 1 \quad .. \quad .. \quad .. \quad (e)$$

We now denote

$$S = (\vec{E} \cdot \vec{H}) \quad .. \quad .. \quad .. \quad .. \quad .. \quad (f)$$

and obtain for S from (d)

$$S = \frac{G\,(1 - \vec{E}^2)}{\sqrt{1 + \vec{B}^2 - \vec{E}^2 - G^2}} \quad .. \quad .. \quad .. \quad (g)$$

From (g) and (e) it follows

$$V = \frac{S}{G} - 1 \quad .. \quad .. \quad .. \quad .. \quad .. \quad (h)$$

Now we have only to express G in terms of $\vec{E}$ and $\vec{H}$. For this purpose we

[7] In this paragraph, we have followed exactly the same method as in the appendix, due to P. Weiss in Born–Infeld, Ref. (1), p. 544.

write first (*d*) in the following form

$$\vec{H} = \frac{\vec{B} - G\vec{E}}{1 - \vec{E}^2} \cdot \frac{1 - \vec{E}^2}{\sqrt{1 + \vec{B}^2 - \vec{E}^2 - G^2}}$$

and because of (*g*)

$$\vec{H} = \frac{\vec{B} - G\vec{E}}{1 - \vec{E}^2} \cdot \frac{S}{G} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

Now we solve (*g*) and (*i*) with respect to G by eliminating $\vec{B}$.

Squaring (*g*) gives

$$- S^2 \vec{B}^2 + G^2 \{S^2 + (1 - \vec{E}^2)^2\} = S^2 (1 - \vec{E}^2) \quad \ldots \quad (j)$$

Squaring (*i*) gives

$$S^2 \vec{B}^2 = G^2 \{\vec{H}^2 (1 - \vec{E}^2)^2 + S^2 (2 - \vec{E}^2)\} \quad \ldots \quad \ldots \quad (k)$$

From (*j*) and (*k*) follows

$$G^2 \{(1 - \vec{E}^2)^2 (1 - \vec{H}^2) - S^2 (1 - \vec{E}^2)\} = S^2 (1 - \vec{E}^2)$$

or,

$$G = \frac{S}{\sqrt{(1 - \vec{E}^2)(1 - \vec{H}^2) - S^2}} \quad \ldots \quad \ldots \quad \ldots \quad (l)$$

Introduction of (*l*) into (*h*) gives

$$V = \sqrt{(1 - \vec{E}^2)(1 - \vec{H}^2) - S^2} - 1 \quad \ldots \quad \ldots \quad (m)$$

As regards the equations to be derived from V, we have

$$dV = dL - \vec{B} d\vec{H} - \vec{H} d\vec{B}$$

and

$$dL = \frac{\partial L}{\partial \vec{E}} d\vec{E} + \frac{\partial L}{\partial \vec{B}} d\vec{B}$$

$$= - \vec{D} d\vec{E} + \vec{H} d\vec{B}$$

$$\therefore \; dV = - \vec{D} d\vec{E} + \vec{H} d\vec{B} - \vec{B} d\vec{H} - \vec{H} d\vec{B}$$

$$= - \vec{D} d\vec{E} - \vec{B} d\vec{H}$$

$$\therefore \; \vec{D} = - \frac{\partial V}{\partial \vec{E}}; \; \vec{B} = - \frac{\partial V}{\partial \vec{H}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Using for V the expression obtained in (*m*) above, these can be written as

$$\left.\begin{aligned}\vec{D} &= \frac{(1-\vec{H}^2)\,\vec{E}+S\vec{H}}{\sqrt{(1-\vec{E}^2)\,(1-\vec{H}^2)-S^2}}\\ \vec{B} &= \frac{(1-\vec{E}^2)\,\vec{H}+S\vec{E}}{\sqrt{(1-\vec{E}^2)\,(1-\vec{H}^2)-S^2}}\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

9. Lorentz-invariance of the field equations derived from V.

We now proceed to prove that the equations (16) are invariant with respect to the (C)-transformations contained in (6′).

We write

$$\vec{E}'^2 = \vec{E}^2 - 2i\delta\vec{E}\,(\vec{\beta} \times \overrightarrow{B + iE}) = \vec{E}^2 - 2i\delta\vec{E}\,(\beta \times \vec{B}) = \vec{E}^2 - 2i\delta\vec{\beta}\,(\vec{B} \times \vec{E})$$

$$\vec{H}'^2 = \vec{H}^2 + 2\delta\vec{H}\,(\vec{\beta} \times \overrightarrow{H + iD}) = \vec{H}^2 + 2i\delta\vec{H}\,(\beta \times \vec{D}) = \vec{H}^2 - 2i\delta\vec{\beta}\,(\vec{H} \times \vec{D})$$

$$\begin{aligned}S' = (\vec{E}'\cdot\vec{H}') &= S + \delta\vec{E}\,(\vec{\beta} \times \overrightarrow{H + iD}) - i\delta\vec{H}\,(\vec{\beta} \times \overrightarrow{B + iE})\\ &= S + i\delta\vec{E}\,(\vec{\beta} \times \vec{D}) - i\delta\vec{H}\,(\vec{\beta} \times \vec{B}) = S - i\delta\vec{\beta}\,(\vec{E} \times \vec{D})\\ &\qquad\qquad + i\delta\vec{\beta}\,(\vec{H} \times \vec{B})\\ &= S + 2i\delta\vec{\beta}\,(\vec{H} \times \vec{B}).\end{aligned}$$

$$S'^2 = S^2 + 4i\delta S\vec{\beta}\,(\vec{H} \times \vec{B})$$

$$\therefore \quad (1 - \vec{E}'^2)\,(1 - \vec{H}'^2) - S'^2 = \Omega + 2i\delta\Omega_2$$

where $\Omega = (1 - \vec{E}^2)\,(1 - \vec{H}^2) - S^2$

and $\Omega_2 = (1 - \vec{E}^2)\,\vec{\beta}\,(\vec{H} \times \vec{D}) + (1 - \vec{H}^2)\,\vec{\beta}\,(\vec{B} \times \vec{E}) - 2S\vec{\beta}\,(\vec{H} \times \vec{B}).$

Similarly $\vec{E}'(1 - \vec{H}'^2) + \vec{H}'S' = \left\{\vec{E}\,(1 - \vec{H}^2) + \vec{H}\,S\right\} + \delta\vec{\Omega}_1$, where

$$\begin{aligned}\vec{\Omega}_1 = 2i\vec{E}\,(\vec{\beta}\vec{H}\vec{D}) - i\,(1 - \vec{H}^2)\,(\vec{\beta} \times \overrightarrow{B + iE}) + 2i\vec{H}\,(\vec{\beta}\vec{H}\vec{B})\\ + S\,(\vec{\beta} \times \overrightarrow{H + iD}).\end{aligned}$$

Hence the right-hand side of the first equation in (16) in the co-ordinate

system Σ' reduces to

$$\frac{\{\vec{E}\,(1-\vec{H}^2)+\vec{H}S\}+\delta\vec{\Omega}_1}{\sqrt{\Omega+2i\delta\Omega_2}} = \vec{D} + \frac{\delta\vec{\Omega}_1}{\sqrt{\Omega}} - \frac{i\delta\Omega_2\vec{D}}{\Omega}, \text{ just as (11).}$$

Corresponding to (12) and (13), Ω_2 and $\vec{\Omega}_1$ simplify to

$$\Omega_2 = 2\sqrt{\Omega}\,\vec{\beta}\,(\vec{H}\times\vec{E})$$

and $\vec{\Omega}_1 = \sqrt{\Omega}\,(\vec{\beta}\times\vec{D}) - i\sqrt{\Omega}\,(\vec{\beta}\times\vec{H}) + 2i\vec{E}\,(\vec{\beta}\vec{H}\vec{D}) + 2i\vec{H}\,(\vec{\beta}\vec{H}\vec{B})$.

Hence

$$\frac{\vec{E}'\,(1-\vec{H}'^2)+\vec{H}'S'}{\sqrt{(1-\vec{E}'^2)\,(1-\vec{H}'^2)-S'^2}} = \vec{D} + \delta\,(\vec{\beta}\times\vec{D}) - i\delta\,(\vec{\beta}\times\vec{H})$$

$$+ \frac{2i\delta}{\sqrt{\Omega}}\left[\vec{H}\,(\vec{\beta}\vec{H}\vec{B}) + \vec{E}\,(\vec{\beta}\vec{H}\vec{D}) - \vec{D}\,(\vec{\beta}\vec{H}\vec{E})\right]$$

$$= \vec{D} + \delta\,(\vec{\beta}\times\vec{D}) - i\delta\,(\vec{\beta}\times\vec{H}) + \frac{2i\delta}{\sqrt{\Omega}}\left\{\vec{\beta}\,(\vec{E}\vec{H}\vec{D})\right\}$$

[using (8) and the vector-identity],

$$= \vec{D} - i\delta\,(\vec{\beta}\times\overrightarrow{H+iD}), \text{ since } (\vec{E}\vec{H}\vec{D}) = (\vec{H}\vec{D}\vec{E})$$

$$= (\vec{H}\vec{H}\vec{B}) \text{ from (8)} = 0.$$

$$= \vec{D}', \text{ using } (6').$$

This proves the Lorentz-invariance of the first equation of (16).

For the second equation, we need only calculate $\vec{\Omega}_3$ corresponding to (13′). This is easily found to be

$$\vec{\Omega}_3 = \sqrt{\Omega}\,(\vec{\beta}\times\vec{B}) + i\sqrt{\Omega}\,(\vec{\beta}\times\vec{E}) + 2i\vec{H}\,(\vec{\beta}\vec{B}\vec{E}) + 2i\vec{E}\,(\vec{\beta}\vec{H}\vec{B})$$

and we find, as in the previous case,

$$\frac{(1-\vec{E}'^2)\,\vec{H}'+\vec{E}'S'}{\sqrt{(1-\vec{E}'^2)\,(1-\vec{H}'^2)-S'^2}} = \vec{B} + \delta\,(\vec{\beta}\times\vec{B}) + i\delta\,(\vec{\beta}\times\vec{E})$$

$$= \vec{B} + \delta\,(\vec{\beta}\times\overrightarrow{B+iE})$$

$$= \vec{B}',$$

proving the invariance of the second of the equations (16).

10. Conclusion.

We have proved the Lorentz-invariance of the equations connecting the primary and secondary vectors in the cases where the action functions are U and V, and the Lorentz-invariance of the field equations derived U and V follows as an immediate consequence.

I wish to thank Prof. Born for proposing the problem and suggesting that I should use the Einstein-Mayer semi-vector theory for proving the Lorentz-invariance of the equations concerned.

P.T.O

(5)

A THEOREM ON ACTION-FUNCTIONS IN

BORN'S FIELD THEORY.

Reprinted from "The Proceedings of the Indian Academy of Sciences", Vol. IV, No. 3, Sec. A, 1936.

A THEOREM ON ACTION FUNCTIONS IN BORN'S FIELD THEORY.

BY B. S. MADHAVA RAO.

(University of Mysore)

(From the Department of Physics, Indian Institute of Science, Bangalore.)

Received August 19, 1936.

1. *Introduction.*

INFELD has shown[1] that, in the development of Born's field theory from a variational principle, it is possible to replace the action function originally adopted by Born and Infeld[2] by a more general one T which satisfies the conditions

$$p^{kl} = \frac{\partial \mathrm{T}}{\partial f_{kl}}\,;\; f^{*kl} = \frac{\partial \mathrm{T}}{\partial p_{kl}^{*}} \qquad \ldots \quad (1)$$

Here, T is assumed a function of F, P, R, where

$$\mathrm{F} = \tfrac{1}{2} f_{kl} f^{kl}\,;\; \mathrm{P} = \tfrac{1}{2} p_{kl}^{*} p^{*kl},\; \mathrm{R} = -\tfrac{1}{2} f^{*}_{kl} p^{*kl} = \tfrac{1}{2} f_{kl} p^{kl} \qquad \ldots \quad (2)$$

and f_{kl} as well as p_{kl}^{*} are treated as independent variables. We shall call action functions satisfying conditions (1) *self-conjugate action functions.*

Infeld has deduced the necessary and sufficient conditions for an action function being self-conjugate and used these conditions to obtain a new action function which leads, in the static case, to two solutions with central symmetry one giving a finite, the other an infinite energy. There is however a fundamental difference between Infeld's action function and the Lagrangian used by Born and Infeld in that while the latter is derived from considerations of relativistic invariance the former is not.

In trying to construct action functions which satisfy both the conditions of self-conjugacy and relativistic invariance, I have come to the conclusion that there are no such functions other than Born's action function. I have been thus led to prove in this paper the

THEOREM :—*The only self-conjugate action function which satisfies the condition of relativistic invariance is Born's action function.*

2. *Infeld's Conditions for Self-Conjugate Action Functions.*

Infeld has shown that the condition of T being self-conjugate and the assumption that T = Lagrangian + Hamiltonian are entirely equivalent.

[1] Infeld, *Proc. Camb. Phil. Soc.*, 1936, 32, Part I, p. 127.

[2] Born and Infeld, *Proc. Roy. Soc.*, A, 1934, **144**, 425.

Another form of the conditions he has deduced is that of the equations

$$R = T_F \cdot F - T_P \cdot P \qquad \ldots \qquad (3)$$

$$R^2 = -FP \qquad \ldots \qquad (4)$$

$$T_F \cdot F + T_P \cdot P + T_R \cdot R = 0 \qquad \ldots \qquad (5)$$

the independence of *only two* is a necessary and sufficient condition for T being self-conjugate. This is always the case if T can be represented as a homogeneous function of F, P, R of zero degree. In this case, (5) is identically satisfied and the other two determine F and R as functions of P.

3. *Born's Action Function.*

In the form given by Infeld, Born's action function is

$$T = \frac{F - P}{R} - 2 \qquad \ldots \qquad (6)$$

This satisfies the condition (5) and also the condition that in the limiting case for $f_{kl} = p_{kl}$ and when f_{kl} and p_{ki} are very small, T is equal to zero as it ought to be since in this case the Maxwellian field equations represent the limiting case for very weak fields.

We shall derive the form (6) so as to bring out explicitly the notion of relativistic invariance. The variation principle of least action is to be used in the form

$$\delta \int T d\tau = 0 \quad (d\tau = dx^1 dx^2 dx^3 dx^4) \qquad \ldots \qquad (7)$$

and T is to be found from the postulate that the *action integral has to be an invariant.* In this case, the field is determined by the two covariant tensors a_{kl} and b_{kl} which could be split up into symmetrical and antisymmetrical parts by

$$\left.\begin{aligned} a_{kl} = g_{kl} + f_{kl}\,;\ b_{kl} = g_{kl} + p_{kl}^* \\ g_{kl} = g_{lk}\,;\ f_{kl} = -f_{lk}\,;\ p_{kl}^* = -p_{lk}^* \end{aligned}\right\} \qquad \ldots \qquad (8)$$

The reason for assuming that the field is determined by two covariant tensors is that, in consonance with Infeld, we assume f_{kl} and p_{kl}^* to be independent variables. The invariance of the action-integral leads to the condition that T should be any homogeneous function of the determinants of the covariant tensors of order $\frac{1}{2}$[3]. We have the expressions

$$\sqrt{-|g_{kl} + f_{kl}|},\ \sqrt{-|g_{kl} + p_{kl}^*|};\ \sqrt{-|g_{kl}|},\ \sqrt{|f_{kl}|};\ \sqrt{|p_{kl}^*|} \ldots \quad (9)$$

which multiplied by $d\tau$ are invariant, where the minus sign is added in order to get real values of the square roots.

[3] Born and Infeld, *Proc. Roy. Soc.*, A, 1934, **144**, 429.

The simplest homogeneous function of degree $\frac{1}{2}$ which we can form out of the expressions (9) is the linear expression

$$T = \sqrt{-|g_{kl}+f_{kl}|} + A\sqrt{-|g_{kl}+p_{kl}^*|} + B\sqrt{-|g_{kl}|} + C\sqrt{|f_{kl}|} + D\sqrt{|p_{kl}^*|} \quad .. \quad (10)$$

Assuming that f_{kl} and p_{kl} are rotations of potential vectors, we can take $C = 0$, $D = 0$. Further A and B are determined by the condition that in the limiting case of Cartesian Co-ordinates and weak fields, T must reduce to $\frac{1}{2}$ (F + P) so that T may be equal to zero for $p_{kl} = f_{kl}$.

Now,

$$\left.\begin{aligned} -|g_{kl}+f_{kl}| &= 1 + F - |f_{kl}| = 1 + F - G^2 \\ -|g_{kl}+p_{kl}^*| &= 1 + P - |p_{kl}^*| = 1 + P - Q^2 \end{aligned}\right\} \quad .. \quad (11)$$

where $G = \frac{1}{4} f_{kl} f^{*kl}$, and $Q = \frac{1}{4} p^{kl} p_{kl}^*$

For small values of f_{kl} and p_{kl}^* the last determinants on the right-hand sides of the first two expressions of (11) can be neglected and (10) becomes equal to $\frac{1}{2}$ (F + P) only if $A = 1$, and $B = -2$. Hence (10) reduces to

$$T = \sqrt{1+F} + \sqrt{1+P} - 2 \quad .. \quad .. \quad .. \quad .. \quad (12)^{4}$$

where, for the sake of simplicity, we have neglected G and Q, thus assuming that T depends only on F and P.

Applying equations (3) and (5) to (12), we have

$$R = \frac{F}{2\sqrt{1+F}} - \frac{P}{2\sqrt{1+P}} \quad .. \quad .. \quad .. \quad .. \quad (13)$$

$$0 = \frac{F}{2\sqrt{1+F}} + \frac{P}{2\sqrt{1+P}} \quad .. \quad .. \quad .. \quad .. \quad (14)$$

From (13) and (14) we get

$$\sqrt{1+F} = \frac{F}{R}, \text{ and } \sqrt{1+P} = -\frac{P}{R}$$

and (12) reduces to

$$T = \frac{F-P}{R} - 2$$

which is expression (6). This derivation shows that Born's T is self-conjugate and makes the action integral relativistic invariant.

4. *Proof of the Theorem.*

Instead of the linear expression (10) let us consider the general homogeneous function of degree $\frac{1}{2}$ which can be formed out of $-|g_{kl}+f_{kl}|$:

[4] This form of (12) is an immediate deduction from Infeld's theorem that T is to be the sum of a Lagrangian and a Hamiltonian for T being self-conjugate.

$-\,|\,g_{kl} + p_{kl}^*\,|$ and $-\,|\,g_{kl}\,|$, neglecting again the dependence of T on the invariants G and Q. We have, therefore, to form the general homogeneous function of order unity in the variables

$$\sqrt{-|\,g_{kl} + f_{kl}\,|},\ \sqrt{-|\,g_{kl} + p_{kl}^*\,|} \text{ and } \sqrt{-|\,g_{kl}\,|}$$

Since the third term is a numerical invariant equal to unity, such a general function can be written, using (11), in the form

$$\mathrm{T} = f\,(\sqrt{1+\mathrm{F}},\ \sqrt{1+\mathrm{P}}) + \mathrm{A}\,\sqrt{-\,|\,g_{kl}\,|} \qquad \ldots \qquad \ldots (15)$$

where f *is a homogeneous function of order unity in* $\sqrt{1+\mathrm{F}}$ and $\sqrt{1+\mathrm{P}}$. Putting $\sqrt{1+\mathrm{F}} = \alpha$, and $\sqrt{1+\mathrm{P}} = \beta$, (15) can be written as

$$\mathrm{T} = f\,(\alpha, \beta) + \mathrm{A} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots (16)$$

The constant A in (16) is to be determined from the condition that in the limiting case of weak fields $\mathrm{T} \rightarrow \frac{1}{2}(\mathrm{F}+\mathrm{P})$. Writing

$$f\,(\alpha, \beta) = \phi\,(\mathrm{F}, \mathrm{P}),$$

$$\phi\,(\mathrm{F}, \mathrm{P}) = \phi\,(0, 0) + \left\{\mathrm{F}\left(\frac{\partial\phi}{\partial \mathrm{F}}\right)_{0,0} + \mathrm{P}\left(\frac{\partial\phi}{\partial \mathrm{P}}\right)_{0,0}\right\} + \text{etc.}$$

$$= f\,(1, 1) + \left\{\mathrm{F}\left(\frac{\partial f}{\partial \alpha}\right)_{1,1}\left(\frac{\partial \alpha}{\partial \mathrm{F}}\right)_{0,0} + \mathrm{P}\left(\frac{\partial f}{\partial \beta}\right)_{1,1}\left(\frac{\partial \beta}{\partial \mathrm{P}}\right)_{0,0}\right\} + \text{etc.}$$

$$= f\,(1, 1) + \tfrac{1}{2}\left\{\mathrm{F}\left(\frac{\partial f}{\partial \alpha}\right)_{1,1} + \mathrm{P}\left(\frac{\partial f}{\partial \beta}\right)_{1,1}\right\} + \quad \ldots \qquad \ldots (17)$$

Since $f\,(\alpha, \beta)$ is homogeneous of order one in α and β

$$\alpha\frac{\partial f}{\partial \alpha} + \beta\frac{\partial f}{\partial \beta} = f \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots (18)$$

$$\therefore \quad \left(\frac{\partial f}{\partial \alpha}\right)_{1,1} + \left(\frac{\partial f}{\partial \beta}\right)_{1,1} = f\,(1, 1) \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots (19)$$

In order that $\mathrm{T} \longrightarrow \frac{1}{2}(\mathrm{F}+\mathrm{P})$ as α and $\beta \longrightarrow'$, (17) shows that

$$\left(\frac{\partial f}{\partial \alpha}\right)_{1,1} = \left(\frac{\partial f}{\partial \beta}\right)_{1,1} = 1$$

and this gives, taking (19) into consideration

$$f\,(1, 1) = 2$$

and therefore $\mathrm{A} = -\,2$. Thus the general action function satisfying the condition of relativistic invariance and the limiting condition of reduction to the Maxwellian case is

$$\mathrm{T} = f\,(\alpha, \beta) - 2 \quad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots (20)$$

We will now determine the form of f by applying Infeld's conditions for being self-conjugate. From the equations (3) and (5)

$$\left.\begin{aligned} \mathrm{R} &= \mathrm{T}_{\mathrm{F}}\cdot\mathrm{F} - \mathrm{T}_{\mathrm{P}}\cdot\mathrm{P} \\ 0 &= \mathrm{T}_{\mathrm{F}}\cdot\mathrm{F} + \mathrm{T}_{\mathrm{P}}\cdot\mathrm{P} \end{aligned}\right\}$$

or

$$\left.\begin{aligned} 2T_F &= \frac{R}{F} \\ \text{and}\qquad 2T_P &= -\frac{R}{P} \end{aligned}\right\} \qquad \ldots \qquad (21)$$

$$\left.\begin{aligned} f_\alpha &= T_\alpha = T_F \cdot \frac{\partial F}{\partial \alpha} = T_F \cdot 2\alpha \\ f_\beta &= T_\beta = T_P \cdot \frac{\partial P}{\partial \beta} = T_P \cdot 2\beta \end{aligned}\right\} \text{ since } \alpha = \sqrt{1+F},\ \beta = \sqrt{1+P}$$

Hence from (21),

$$\left.\begin{aligned} f_\alpha &= \frac{R}{F}\alpha \\ \text{and}\qquad f_\beta &= -\frac{R}{P}\beta \end{aligned}\right\}$$

Substituting these values of f_α and f_β in equation (18) *viz.*,

$$\alpha f_\alpha + \beta f_\beta = f(\alpha, \beta)$$

which is valid on account of the homogeneity of f, we get

$$\begin{aligned} f(\alpha, \beta) &= \frac{R}{F}\alpha^2 - \frac{R}{F}\beta^2 \\ &= \frac{R(1+F)}{F} - \frac{R(1+P)}{P} \\ &= \frac{R}{FP}\left\{P(1+F) - F(1+P)\right\} \\ &= \frac{R(P-F)}{FP} \end{aligned}$$

Using, now, the relation (4), the right-hand side becomes $(F - P)/R$.

$$\therefore\quad f(\alpha, \beta) = \frac{F-P}{R} \text{ and (20) reduces to}$$

$$T = \frac{F-P}{R} - 2$$

which is the same as Born's action function (6). We have, therefore, proved the theorem enunciated in § 1.

5. *Conclusion.*

I wish to thank Prof. Max Born for suggesting this problem during the course of his lectures on the new field theory which I had the privilege of attending last year at the Indian Institute of Science, Bangalore.

P.T.O

(6)

GENERALISED ACTION-FUNCTIONS IN BORN'S ELECTRO-DYNAMICS.

--oOo--

(This paper will be published in the Proc.Ind. Acad.Sc. A. 1937, Vol.6, September 37).

1. Introduction.

Infeld[1] has shown that, in the development of Born's field theory, it is possible to choose an infinity of action-functions other than the one originally proposed by Born and Infeld[2] which, for simplicity, might be called Born's action function. These action-functions of Infeld and the one recently introduced by Hoffmann and Infeld[3] have, in common with Born's action - function, the properties of self-conjugacy, of the existence of simple algebraic relations between the f_{kl} and p_{kl} - fields, and of giving finite values for the energy of the electrical particle. They however differ from the latter in that the condition of invariance of the action integral itself is not insisted in them. As I have shown elsewhere[4] Born's action-function is unique if this condition as well as the condition of self-conjugacy are imposed on the action-function.

Laying aside for the present this condition of

---

(1) Infeld; Proc.Camb.Phil.Soc.1936,32,127 and 1937,33,70 - hereinafter referred to as I and II

(2) Born and Infeld; Proc.Roy.Soc.A.1934;144, 425.

(3) Hoffmann and Infeld; Phys.Rev.1937,51,765 - referred to as III.

(4) B.S.Madhava Rao;Proc.Ind.Acad.Sci.A.1936,3,377

invariance of the action integral, we might say that these action-functions give rise to several systems of Born's non-linear electro-dynamics. These action - functions, however, are all confined to the case where the invariant $G = (\vec{B}\vec{E})$ is neglected as also the invariants $Q = (\vec{D}\vec{H})$ and $S = (\vec{B}\vec{D}) + (\vec{E}\vec{H})$. I have considered in this paper the general case where these invariants are not neglected, and have constructed a two-fold infinity of action-functions all having the same properties as Infeld's functions and reducing to them when $G = 0$. This generalisation has also enabled the deduction of close connections with the complex formalism developed by Weiss(5). In view of the fact, however, that the condition of "finite self-energy" has so far been discussed in the case of the point - singularity, and in the case of the ring-singularity in the zero-approximation(6) where, in both cases, the invariant G does not appear, it seems difficult to set up a criterion limiting the choice of these functions. Nor is the regularity condition of Hoffmann and Infeld of any use since it is based on the character of a

---

(5) P. Weiss; Proc. Camb. Phil. Soc., 1937, 33, p. 79 - referred to as IV.

(6) B.S. Madhava Rao; Proc. Ind. Acad. Sci., A. 1936, 4, 355

spherically-symmetric electrostatic solution.

An attempt has been made in this paper to construct an action-function which leads to a full coincidence of the second order terms when the Lagrangian of the field theory is compared with the Lagrangian that arises from the investigations of Euler and Köckel[7] on scattering of light by light on the basis of Dirac's theory of holes. It is well - known that such a coincidence does not exist[8] in the terms containing G^2 when Born's action function is used. The result that I obtain is that even this condition does not restrict the choice, there being more than one such function wherefrom we could obtain the coincidence with the Euler - Kockel Lagrangian as well as the good determination of the fine-structure constant possible with the Hoffmann-Infeld action function.

I have closely followed the method adopted by Infeld in I and II.

2. Condition of self - conjugacy.

We introduce the f_{kl} and p^{kl} (or in the dual form f^{*kl} and p^{*}_{kl}) tensors describing the electro-

(7) Euler and Kockel; Naturwiss, 1935,23,246.

(8) M. Born; Proc.Ind.Acad.Sci., A.1935,2, 560.

magnetic field, the space-vector notation, and also the several invariants in the usual way[9] with the modification introduced by Weiss[10] which leads to a good lot of simplification by avoiding unnecessary numerical factors. The Lorentz frame of reference is used

$$\left.\begin{aligned} (f_{23}, f_{31}, f_{12}) &= \vec{B}\,; \quad (f_{14}, f_{24}, f_{34}) = \vec{E} \\ (p_{23}, p_{31}, p_{12}) &= \vec{H}\,; \quad (p_{14}, p_{24}, p_{34}) = \vec{D} \end{aligned}\right\} \qquad (2,1)$$

$$\left.\begin{aligned} \tfrac{1}{4} f_{kl} f^{kl} = -\tfrac{1}{4} f^{*kl} f^{*}_{kl} &= F\,; \quad \tfrac{1}{4} f_{kl} f^{*kl} = \tfrac{1}{4} f^{*}_{kl} f^{kl} = G \\ \tfrac{1}{4} p^{*}_{kl} p^{*kl} = -\tfrac{1}{4} p_{kl} p^{kl} &= P\,; \quad \tfrac{1}{4} p^{*}_{kl} p^{kl} = \tfrac{1}{4} p_{kl} p^{*kl} = Q \\ \tfrac{1}{4} f_{kl} p^{kl} = -\tfrac{1}{4} f^{*kl} p^{*}_{kl} &= R\,; \quad \tfrac{1}{4} f_{kl} p^{*kl} = \tfrac{1}{4} f^{*kl} p_{kl} = S \end{aligned}\right\} \qquad (2,2)$$

$$\left.\begin{aligned} F &= \tfrac{1}{2}(\vec{B}^2 - \vec{E}^2)\,; \quad G = (\vec{B}\vec{E}) \\ P &= \tfrac{1}{2}(\vec{D}^2 - \vec{H}^2)\,; \quad Q = (\vec{D}\vec{H}) \\ R &= \tfrac{1}{2}\{(\vec{B}\vec{H}) - (\vec{D}\vec{E})\}\,; \quad S = \tfrac{1}{2}\{(\vec{B}\vec{D}) + (\vec{E}\vec{H})\} \end{aligned}\right\} \qquad (2,3)$$

We impose on our action function T which we assume to be a function of F, G, P, Q that it should be self-conjugate. This means that the set of equations

---

(9) B.S. Madhava Rao; Proc. Ind. Acad. Sci., A. 1936, 4, 578, 3 -referred to as V.

(10) See IV; footnote on p.80.

$$p^{kl} = \frac{\partial T}{\partial f_{kl}} \qquad (2,4)$$

$$f^{*kl} = \frac{\partial T}{\partial p^{*}_{kl}} \qquad (2,5)$$

should be self-consistent, i.e. that (2,5) is a consequence of (2,4), or vice-versa. We can re-write (2,4) and (2,5) in the form

$$p^{kl} = T_F f^{kl} + T_G f^{*kl} \qquad (2,6)$$

$$f^{*kl} = T_P p^{*kl} + T_Q p^{kl} \qquad (2,7)$$

where T_F, T_G, T_P, T_Q denote the derivatives with respect to F, G, P and Q respectively. We shall now find the restriction to be imposed on   to ensure the consistency of (2,6) and (2,7). We multiply (2,6) & (2,7) respectively by $(\frac{1}{4}f_{kl}, \frac{1}{4}f^{*}_{kl}, \frac{1}{4}p_{kl}, \frac{1}{4}p^{*}_{kl})$; $(\frac{1}{4}p^{*}_{kl}, \frac{1}{4}p_{kl}, \frac{1}{4}f^{*}_{kl}, \frac{1}{4}f_{kl})$ and sum up. We get(11) after suitably grouping the terms

$$R = F T_F + G T_G \qquad (2,8)$$

$$-R = P T_P + Q T_Q \qquad (2,9)$$

$$S = G T_F - F T_G \qquad (2,10)$$

$$S = Q T_P - P T_Q \qquad (2,11)$$

(11) As in V, p.579 where we replace both L and H by T and introduce the necessary numerical factors to correspond to Weiss's notation.

$$-P = R\,T_F + S\,T_G \tag{2,12}$$

$$F = R\,T_P - S\,T_Q \tag{2,13}$$

$$Q = S\,T_F - R\,T_G \tag{2,14}$$

$$G = S\,T_P + R\,T_Q \tag{2,15}$$

The equations (2,9), (2,11), (2,13) and (2,15) must be consequences of (2,8), (2,10), (2,12) and (2,14) respectively.

Before proceeding to reduce the above number of eight equations, we will directly obtain two conditions which follow from the self-consistency of (2,6) and (2,7). Taking the dual of (2,6) we get

$$p^{*kl} = T_F f^{*kl} + T_G f^{**kl} = T_F f^{*kl} - T_G f^{kl}$$

Substituting this value of p^{*kl} and the value of p^{kl} from (2,6) in (2,7) we should get an indentical equation i.e.

$$f^{*kl} = T_P\left(T_F f^{*kl} - T_G f^{kl}\right) + T_Q\left(T_F f^{kl} + T_G f^{*kl}\right)$$

identically. This leads to the consistency conditions

$$T_F T_P + T_G T_Q = 1 \tag{2,16}$$

$$T_F T_Q - T_P T_G = 0 \tag{2,17}$$

It is now easy to see with the aid of (2,16) and (2,17)

that the equations (2,12) - (2,15) are consequences of (2,8) - (2,11). For example from (2,9) and (2,11) we get

$$R T_F + S T_G = -T_F \left(P T_P + Q T_Q \right) + T_G \left(Q T_P - P T_Q \right)$$

$$= -P \left(T_F T_P + T_G T_Q \right) - Q \left(T_F T_Q - T_P T_G \right)$$

$$= -P, \quad \text{from } (2,16), (2,17);$$

But this is the same equation as (2,12). Similarly for the other equations. Hence the eight equations (2,8) -(2,15) can be reduced to the six equations (2,8) - (2,11) and (2,16) - (2,17).

In essence we have replaced the four equations (2,12) - (2,15) by the two equations (2,16) and (2,17). We can now find two other equations equivalent to the latter. Multiplying (2,8) and (2,9), also (2,10) and (2,11) we have

$$-R^2 = FP\, T_F T_P + GQ\, T_G T_Q + FQ\, T_F T_Q + PG\, T_P T_G$$

$$S^2 = GQ\, T_F T_P + FP\, T_G T_Q - PG\, T_F T_Q - FQ\, T_P T_G$$

Adding

$$S^2 - R^2 = \left(T_F T_P + T_G T_Q \right)\left(FP + GQ \right) + \left(T_F T_Q - T_P T_G \right)\left(FQ - PG \right)$$

Or, using (2,16) and (2,17)

$$\left.\begin{aligned} S^2 - R^2 &= FP + GQ \\ \text{Similarly} \quad 2RS &= FQ - PG \end{aligned}\right\} \qquad \begin{matrix} (2,18) \\ (2,19) \end{matrix}$$

relations expressing R and S interms of F, G, P, Q

Further, using (2,8) - (2,11) and (2,18) - (2,19) we can deduce (2,16) and (2,17). In fact, solving (2,8) - (2,11) for the derivatives we get

$$\left.\begin{aligned} T_F &= \frac{FR+GS}{F^2+G^2} \;; \quad T_G = \frac{GR-FS}{F^2+G^2} \\ T_P &= \frac{QS-PR}{P^2+Q^2} \;; \quad T_Q = -\frac{QR+PS}{P^2+Q^2}. \end{aligned}\right\} \qquad (2,20)$$

Using the expressions we can easily derive

$$\left.\begin{aligned} T_F T_P + T_G T_Q &= \frac{(S^2-R^2)(FP+GQ)+2RS(FQ-PG)}{(F^2+G^2)(P^2+Q^2)} \\ T_F T_Q - T_P T_G &= \frac{2RS(FP+GQ)-(S^2-R^2)(FQ-PG)}{(F^2+G^2)(P^2+Q^2)} \end{aligned}\right\}$$

Also from (2,18), (2,19) by squaring and adding

$$R^2+S^2 = \sqrt{(F^2+G^2)(P^2+Q^2)} \qquad (2,21)$$

Using (2,18), (2,19) and (2,21) the above expressions lead at once to the consistency conditions (2,16), (2,17).

Thus the original system of eight equations can be reduced to six, viz., (2,8) - (2,11) and either the two ~~two~~ consistency conditions or Weiss's relations between the invariants. These last are obtained by Weiss[12] by expressing the symmetry of the energy-

(12) See IV; p.85, equation (4,5).

impulse tensor. Thus <u>the relations between the invariants imposed by self-conjugacy is the same as that obtained by making the energy-impulse tensor symmetrical.</u>

Coming now to the conditions (2,8) - (2,11) we can re-write them in the forms

$$\left.\begin{aligned} F\,T_F + G\,T_G + P\,T_P + Q\,T_Q &= 0 \\ G\,T_F - F\,T_G - Q\,T_P + P\,T_Q &= 0 \end{aligned}\right\} \qquad \begin{aligned}(2,22)\\(2,23)\end{aligned}$$

and

$$\left.\begin{aligned} F\,T_F + G\,T_G - P\,T_P - Q\,T_Q &= 2R \\ G\,T_F - F\,T_G + Q\,T_P - P\,T_Q &= 2S \end{aligned}\right\} \qquad \begin{aligned}(2,24)\\(2,25)\end{aligned}$$

(2,22) - (2,23) form a system of simultaneous partial differential equations for T in the homogeneous form - in particular (2,22) shows that T must be a homogeneous function of degree zero in F, G, P, Q. If we therefore choose T so as to satisfy these relations (2,22) - (2,23), we will have reduced our original eight equations to four only in consonance with the consistency condition. The equations (2,22) - (2,23) would then be indentically satisfied, (2,18) - (2,19) or (2,16) - (2,17) would give the relations between the invariants and (2,24) - (2,25) would give P, Q as functions of F and G, or vice versa.

3. Solution of the system of partial differential equations.

In the usual notation of the theory of partial differential equations, we can write (2,22) - (2,23) as

$$\left.\begin{aligned} F_1 &\equiv x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 = 0 \qquad (3,1) \\ F_2 &\equiv x_2 p_1 - x_1 p_2 - x_4 p_3 + x_3 p_4 = 0 \qquad (3,2) \end{aligned}\right\}$$

and proceed to find out the most general functional form of the solution common to (3,1) - (3,2).

It is easy to verify that the Poisson bracket $(F_1, F_2) = 0$ so that the equations form a complete Jacobian system as they are. The most general solution of (3,1) is any arbitrary function of

$(x_2/x_1, x_3/x_1, x_4/x_1)$. Adopting the usual method we introduce the new variables given by

$$x_1 = x_1, \quad u = x_2/x_1, \quad v = x_3/x_1, \quad w = x_4/x_1 .$$

With these substitutions, equation (3,1) i.e.

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} + x_4 \frac{\partial f}{\partial x_4} = 0 \text{ reduces to}$$

$\partial f/\partial x_1 = 0$, and the second equation (3,2) i.e.

$$x_2 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_2} - x_4 \frac{\partial f}{\partial x_3} + x_3 \frac{\partial f}{\partial x_4} = 0,$$

reduces to

$$x_2 \left(\frac{\partial f}{\partial x_1} - \frac{u}{x_1} f_u - \frac{v}{x_1} f_v - \frac{w}{x_1} f_w \right) - x_1 \left(\frac{1}{x_1} f_u \right)$$
$$- x_4 \left(\frac{1}{x_1} f_v \right) + x_3 \left(\frac{1}{x_1} f_w \right) = 0,$$

$$\text{ie}\quad -u^2 f_u - uv f_v - uw f_w - f_u - w f_v + v f_w = 0$$

$$\text{or,}\quad f_u(1+u^2) + f_v(uv+w) + f_w(uw-v) = 0, \qquad (3,3)$$

The Lagrangian subsidiary equations of (3,3) are

$$\frac{du}{1+u^2} = \frac{dv}{uv+w} = \frac{dw}{uw-v} \qquad (3,4)$$

and we need only find two independent integrals of (3,4).

From (3,4) we can immediately deduce the relations

$$\left.\begin{aligned} \frac{u\,du}{1+u^2} &= \frac{v\,dv + w\,dw}{v^2+w^2} \\ \frac{du}{1+u^2} &= \frac{w\,dv - v\,dw}{v^2+w^2} \end{aligned}\right\}$$

These lead to the two integrals

$$\left.\begin{aligned} &\log(1+u^2) = \log(v^2+w^2) + \text{Const} \\ \text{and,}\quad &\tan^{-1}u - \tan^{-1}\left(\frac{v}{w}\right) = \text{Const} \end{aligned}\right\}$$

For reasons to be given immediately, we write

$$\epsilon = \sqrt{\frac{1+u^2}{v^2+w^2}}\ ;\ \epsilon' = \frac{uv+w}{wu-v}, \left(\text{instead of } \frac{wu-v}{uv+w}\right),$$

or, introducing the invariants

$$\epsilon = \sqrt{\frac{F^2+G^2}{P^2+Q^2}}\ ;\ \epsilon' = \frac{FQ+PG}{FP-GQ} \qquad (3,5)$$

The reasons for choosing ϵ and ϵ' in the forms given by (3,5) is that we want to have these

quantities reduce themselves to the corresponding ones when we go over to Infeld's case where G and Q are neglected. If $G = Q = 0$, then $\epsilon' = 0$ and ϵ reduces to $\pm (F/P)$. We therefore postulate that the negative sign is to be taken in order that our value of ϵ should be the square of the corresponding ϵ used in Infeld's investigations. We may also observe that in the limiting Maxwellian case where $G = Q$, and $F = -P$ we have $\epsilon = 1$, and $\epsilon' = 0$.

The most general form of the solution common to (3,1), (3,2) is any arbitrary function $T(\epsilon, \epsilon')$, of ϵ and ϵ'. If T be chosen in this form, (2,22)-(2,23) are identically satisfied. Our generalisation has introduced the additional parameter ϵ' and gives rise to a two-fold infinity of action functions satisfying the condition of self-conjugacy.

4. T as the sum of Lagrangian and Hamiltonian

We will show that, in our general case also, if (2,4)-(2,5) are satisfied the action function T can be represented as the sum of a Lagrangian and a Hamiltonian

Let

$$\left.\begin{aligned} L(F, G) &= \tfrac{1}{2} T + R \\ H(P, Q) &= \tfrac{1}{2} T - R \end{aligned}\right\} \qquad (4,1)$$

(4,2)

where we have put the numerical factor $\frac{1}{2}$ on the right hand sides for T in consonance with the notation of Weiss. From (4,1)

$$2\,dL = T_F\,dF + T_P\,dP + T_G\,dG + T_Q\,dQ + 2\,dR$$

From the relations (2,18) - (2,19)

$$\left.\begin{aligned} 2S\,dS - 2R\,dR &= F\,dP + P\,dF + G\,dQ + Q\,dG \\ 2S\,dR + 2R\,dS &= F\,dQ + Q\,dF - G\,dP - P\,dG \end{aligned}\right\}$$

Eliminating dS from the above equations

$$\begin{aligned} 2\,dR(R^2+S^2) = {} & dF(QS-PR) - dG\,(PS+QR) \\ & - dP\,(GS+FR) + dQ\,(FS-GR). \qquad (4,3) \end{aligned}$$

i.e. using (2,20) - (2,21)

$$2\,dR = \frac{(T_P\,dF + T_Q\,dG)(P^2+Q^2) - (T_F\,dP + T_G\,dQ)(F^2+G^2)}{\sqrt{(F^2+G^2)(P^2+Q^2)}}$$

From (2,16) - (2,17) and (2,22) -(2,3) we can easily derive

$$\frac{T_P}{T_F} = \frac{T_Q}{T_G} = \left(\frac{F^2+G^2}{P^2+Q^2}\right)^{1/2} = \epsilon \qquad (4,4)$$

and reduce the above equation to

$$2\,dR = T_F\,dF + T_G\,dG - T_P\,dP - T_Q\,dQ$$

and

$$2\,dL = 2\,T_F\,dF + 2\,T_G\,dG,$$

ie. $$\frac{\partial L}{\partial F} = \frac{\partial T}{\partial F}\,; \quad \frac{\partial L}{\partial G} = \frac{\partial T}{\partial G}.$$

Hence L is the Lagrangian. We can prove similarly that H is the Hamiltonian.

5. Expressions for the derivatives of

In (2,20) we have obtained expressions for the derivatives of T in terms of all the six invariants, but since we assume T to be a function of only F, G, P, Q we need expressions for the derivatives which do not involve R and S. This can be done by finding R and S from (2,18) and (2,21) and substituting in (2,20). A simpler way is to use (2,12) - (2,15). From these equations, using the consistency conditions (2,16) - (2,17) we get

$$FP - GQ = -(R^2+S^2)(T_F T_P - T_G T_Q)$$

$$= (R^2+S^2)(1-2\epsilon T_F^2) \quad \text{using } (2,16), (4,4).$$

Hence from (4,4) and (2,21)

$$\left.\begin{aligned} 2T_F^2 &= \sqrt{\frac{P^2+Q^2}{F^2+G^2}} - \frac{FP-GQ}{F^2+G^2} \qquad (5,1) \\ 2T_G^2 &= \sqrt{\frac{P^2+Q^2}{F^2+G^2}} + \frac{FP-GQ}{F^2+G^2} \end{aligned}\right\} \qquad (5,2)$$

In an entirely anologous manner or using the equations (4,4) we can derive

$$\left.\begin{aligned} 2T_P^2 &= \sqrt{\frac{F^2+G^2}{P^2+Q^2}} - \frac{FP-GQ}{P^2+Q^2} \\ 2T_Q^2 &= \sqrt{\frac{F^2+G^2}{P^2+Q^2}} + \frac{FP-GQ}{P^2+Q^2} \end{aligned}\right\} \qquad \begin{aligned}(5,3)\\(5,4)\end{aligned}$$

(5,3) and (5,4) follow from (5,1) and (5,2) by interchanging F and G with P and Q, and thus determine P and Q as the same functions of F and G and conversely, guaranteeing in this way the self-consistency of (2,4) and (2,5).

The equations (5,1) - (5,4) express the derivatives in terms of F, G, P, Q. Since the action functions we are considering can be taken as arbitrary functions of ϵ and ϵ', we must be able to obtain T_ϵ and $T_{\epsilon'}$ in terms of the invariants, in order that we might get for every special system of electrodynamics particular relations connecting F, G, P, Q. We proceed to determine these expressions for T_ϵ and $T_{\epsilon'}$. We can express T_F and T_G in terms of T_ϵ and $T_{\epsilon'}$ in the form

$$\left.\begin{aligned} T_F &= T_\epsilon \cdot \epsilon_F + T_{\epsilon'} \cdot \epsilon'_F \\ T_G &= T_\epsilon \cdot \epsilon_G + T_{\epsilon'} \cdot \epsilon'_G \end{aligned}\right\}$$

where ϵ_F for example, stands for $\partial\epsilon/\partial F$. Solving

these for $T_\epsilon, T_{\epsilon'}$

$$\left.\begin{aligned} T_\epsilon\left(\epsilon_F \epsilon'_G - \epsilon_G \epsilon'_F\right) &= \epsilon'_G T_F - \epsilon'_F T_G \\ -T_{\epsilon'}\left(\epsilon_F \epsilon'_G - \epsilon_G \epsilon'_F\right) &= \epsilon_G T_F - \epsilon_F T_G \end{aligned}\right\}$$

From (3,5)

$$\left.\begin{aligned} \epsilon_F &= \frac{1}{\epsilon}\cdot\frac{F}{P^2+Q^2} \;;\quad \epsilon'_F = -\frac{G\left(P^2+Q^2\right)}{(FP-GQ)^2} \\ \epsilon_G &= \frac{1}{\epsilon}\cdot\frac{G}{P^2+Q^2} \;;\quad \epsilon'_G = \frac{F\left(P^2+Q^2\right)}{(FP-GQ)^2} \end{aligned}\right\}$$

Substituting in the above, and simplifying, we get

$$\left.\begin{aligned} \epsilon T_\epsilon &= F T_F + G T_G \\ (1+\epsilon'^2)\, T_{\epsilon'} &= G T_F - F T_G \end{aligned}\right\}$$

Using (2,8) and (2,10) these can be written in the simple forms

$$\epsilon T_\epsilon = R \qquad (5,5)$$

$$(1+\epsilon'^2)\, T_{\epsilon'} = -S \qquad (5,6)$$

From these equations it can be seen that it is the derivatives $T_\epsilon, T_{\epsilon'}$ that are relevant for the connection between F, G, P, Q, since we can assume in virtue of Weiss's conditions (2,18) - (2,19) that R and S are functions of the remaining invariants. This last procedure of finding R and S involves rather complicated square-root expressions, and does not appear

suitable for particular cases. We shall now find a slightly different method of procedure by a change of the parameters ϵ and ϵ'.

6. Introduction of new parameters.

We can simplify considerations to a great extent by introducing two new functions λ and μ of the invariants defined by

$$\left.\begin{aligned} \epsilon &= e^{\lambda}, \text{ or } \lambda = \log \epsilon \qquad (6,1) \\ \epsilon' &= \tan\mu, \text{ or } \mu = \tan^{-1}\epsilon' \qquad (6,2) \end{aligned}\right\}$$

and consider the action-function as any arbitrary function of λ and μ.

$$\left.\begin{aligned} T_\lambda &= T_\epsilon \cdot \frac{\partial \epsilon}{\partial \lambda} = \epsilon\, T_\epsilon \\ T_\mu &= T_{\epsilon'} \cdot \frac{\partial \epsilon'}{\partial \mu} = (1+\epsilon'^2)\, T_{\epsilon'} \end{aligned}\right\}$$

Hence (5,5) - (5,6) reduce to the very simple and striking forms

$$\left.\begin{aligned} T_\lambda &= R \qquad (6,3) \\ T_\mu &= -S \qquad (6,4) \end{aligned}\right\}$$

We can also express (5,1) - (5,2) in terms of these new parameters. Calculating the value of $(FP-GQ)/(F^2+G^2)$ in terms of ϵ and ϵ', we readily find

$$\left(\frac{FP-GQ}{F^2+G^2}\right)^2 = \frac{1}{\epsilon^2(1+\epsilon'^2)}.$$

so that,

$$(FP-GQ)/(F^2+G^2) = \pm 1/\epsilon\sqrt{1+\epsilon'^2}$$

The choice of the $+$ or $-$ sign is decided by going to the limiting case of Infeld's action-functions. In this case, the left-hand side reduces to (P/F), and since we have already assumed (§ 3) that in this limiting case $(\epsilon'=0)$ we should have our $\epsilon = (-F/P)$, we choose the negative sign, and write

$$\frac{FP-GQ}{F^2+G^2} = -\frac{1}{\epsilon\sqrt{1+\epsilon'^2}} \qquad (6,5)$$

Hence (5,1) - (5,2) reduce to

$$\left.\begin{aligned} 2T_F^2 &= \frac{1}{\epsilon} + \frac{1}{\epsilon\sqrt{1+\epsilon'^2}} \\ 2T_G^2 &= \frac{1}{\epsilon} - \frac{1}{\epsilon\sqrt{1+\epsilon'^2}} \end{aligned}\right\}$$

Or, with the substitutions (6,1) - (6,2) to

$$\left.\begin{aligned} 2T_F^2 &= 2e^{-\lambda}\cos^2\tfrac{1}{2}\mu \\ 2T_G^2 &= 2e^{-\lambda}\sin^2\tfrac{1}{2}\mu \end{aligned}\right\}$$

or,

$$\left.\begin{aligned} T_F &= e^{-\lambda/2}\cos\tfrac{1}{2}\mu & (6,6) \\ T_G &= e^{-\lambda/2}\sin\tfrac{1}{2}\mu & (6,7) \end{aligned}\right\}$$

Making use of (4,4)

$$\left.\begin{aligned} T_P &= e^{\lambda/2}\cos\tfrac{1}{2}\mu & (6,6a) \\ T_Q &= e^{\lambda/2}\sin\tfrac{1}{2}\mu & (6,7a) \end{aligned}\right\}$$

Substituting (6,6)- (6,7) and (6,3) - (6,4) in (2,12) - (2,15) we get

$$\left.\begin{aligned} -P &= e^{-\lambda/2}\left(T_\lambda \cos\tfrac{1}{2}\mu - T_\mu \sin\tfrac{1}{2}\mu\right) \\ F &= e^{\lambda/2}\left(T_\lambda \cos\tfrac{1}{2}\mu + T_\mu \sin\tfrac{1}{2}\mu\right) \end{aligned}\right\} \qquad (6,8)$$

and,

$$\left.\begin{aligned} Q &= -e^{-\lambda/2}\left(T_\mu \cos\tfrac{1}{2}\mu + T_\lambda \sin\tfrac{1}{2}\mu\right) \\ G &= -e^{\lambda/2}\left(T_\mu \cos\tfrac{1}{2}\mu - T_\lambda \sin\tfrac{1}{2}\mu\right) \end{aligned}\right\} \qquad (6,9)$$

Eliminating λ and μ between the four equations (6,8) - (6,9) we get two relations between the invariants F, G, P, Q.

7. Relations with the complex formalism of Weiss.

The parameters λ and μ are very closely connected with the complex invariants

$$\left.\begin{aligned} \phi &= F + iG \\ \psi &= P - iQ \end{aligned}\right\}$$

introduced by Weiss[13] Forming the complex parameter $\lambda + i\mu$

$$\begin{aligned} \lambda + i\mu &= \log \epsilon + i \tan^{-1}\epsilon \\ &= \log\sqrt{F^2+G^2} - \log\sqrt{P^2+Q^2} + i\tan^{-1}\left\{\frac{(G/F) + (Q/P)}{1-(G/F)(Q/P)}\right\} \end{aligned}$$

(13) IV; p.86., equations (5,7) - (5,8).

$$= \left\{ \log\sqrt{F^2+G^2} + i\tan^{-1}\left(\frac{G}{F}\right) \right\} - \left\{ \log\sqrt{P^2+Q^2} - i\tan^{-1}\left(\frac{Q}{P}\right) \right\}$$

$$= \log(F+iG) - \log(P-iQ)$$

$$= \log(\phi/\psi)$$

$$\text{ie} \quad (\phi/\psi) = e^{\lambda+i\mu} \qquad (7,1)$$

From the above equation we see that $\lambda+i\mu$ is an analytic function of both the complex variables ϕ and ψ; hence the Cauchy-Riemann conditions give

$$\left.\begin{aligned} \frac{\partial\lambda}{\partial F} &= \frac{\partial\mu}{\partial G}\,; & \frac{\partial\lambda}{\partial G} &= -\frac{\partial\mu}{\partial F} \\ \frac{\partial\lambda}{\partial P} &= -\frac{\partial\mu}{\partial Q}\,; & \frac{\partial\lambda}{\partial Q} &= \frac{\partial\mu}{\partial P} \end{aligned}\right\} \qquad (7,2)$$

Differentiating (6,6) and (6,7) respectively with respect to F and G

$$\left.\begin{aligned} \frac{\partial^2 T}{\partial F^2} &= -\frac{1}{2}T_F\frac{\partial\lambda}{\partial F} - \frac{1}{2}T_G\frac{\partial\mu}{\partial F} \\ \frac{\partial^2 T}{\partial G^2} &= -\frac{1}{2}T_G\frac{\partial\lambda}{\partial G} + \frac{1}{2}T_F\frac{\partial\mu}{\partial G} \end{aligned}\right\}$$

Adding these equations, and using (7,2) we get the Laplacian equation

$$\left.\begin{aligned} &\frac{\partial^2 T}{\partial F^2} + \frac{\partial^2 T}{\partial G^2} = 0 \\ \text{Similarly} \quad &\frac{\partial^2 T}{\partial P^2} + \frac{\partial^2 T}{\partial Q^2} = 0 \end{aligned}\right\} \qquad (7,3)$$

These equations reduce to the Laplacian equations given by Weiss for L and H (See IV., p.88, equation (6,3)) if ~~we~~ observe that, in accordance with our results in § 4,

$$T = L(F, G) + H(P, Q).$$

8. The Maxwellian case.

This corresponds to $\epsilon = 1, \epsilon' = 0, \lambda = 0, \mu = 0$.
Also we have $F = -P = R$, and $G = Q = S$.
From (6,6) - (6,7)

$$T_F = T_P = 1, \quad \text{and} \quad T_G = T_Q = 0$$

Hence $T = F + P = 0$

9. Born's action function.

The action function $T(\lambda, \mu)$ corresponding to Born's case can be found by using the fact that one of the relations between invariants in this case is $G = Q$
Hence from (2,14) - (2,15)

$$S T_F - R T_G = S T_F + R T_Q$$

$$\text{i.e.} \quad \frac{R}{S} = \frac{1-\epsilon}{1+\epsilon}\frac{T_F}{T_G} \qquad \text{using } (4,4)$$

~~(8,1)~~ i.e. $$\frac{T_\lambda}{T_\mu} = -\tanh\tfrac{1}{2}\lambda \cot\tfrac{1}{2}\mu \qquad (9,1)$$

using (6,1) - (6,4), (6,6) - (6,7)

Solving this partial differential equation we easily deduce

$$T = \frac{\cosh\frac{1}{2}\lambda}{\cos\frac{1}{2}\mu}$$

as the simplest function satisfying (9,1). We therefore write Born's action function in the form

$$T = \frac{2\cosh\frac{1}{2}\lambda}{\cos\frac{1}{2}\mu} - 2 \tag{9,2}$$

so that in the limiting case $\mu = 0$, this might reduce to Infeld's form.(14) Also

$$\left.\begin{aligned} R &= T_\lambda = \sinh\tfrac{1}{2}\lambda \,/\, \cos\tfrac{1}{2}\mu \\ -S &= T_\mu = \left(\cosh\tfrac{1}{2}\lambda \sin\tfrac{1}{2}\mu\right) / \cos^2\tfrac{1}{2}\mu \end{aligned}\right\}$$

Substituting these values in (6,8)-(6,9), we get

$$\left.\begin{aligned} P &= \frac{e^{-\lambda} - \cos\mu}{1 + \cos\mu} \\ F &= \frac{e^{\lambda} - \cos\mu}{1 + \cos\mu} \\ G = Q &= -\tan\tfrac{1}{2}\mu \end{aligned}\right\} \tag{9,3}$$

From the above equations we can easily deduce

$$\left.\begin{aligned} e^{\lambda} &= \frac{1 + 2F - G^2}{1 + G^2} \\ e^{-\lambda} &= \frac{1 + 2P - Q^2}{1 + Q^2} \end{aligned}\right\} \tag{9,4}$$

so that,

$$\frac{1+2F-G^2}{1+G^2} = \frac{1+Q^2}{1+2P-Q^2} \tag{9,5}$$

which, besides $G = Q$, is the other relation between the invariants in "Born's field theory."[15]

A much simpler method of deducing (9,5) is from the 3rd equation of (9,3) itself, for,

$$\tan\mu = \frac{2\tan\frac{1}{2}\mu}{1-\tan^2\frac{1}{2}\mu} = \frac{2G}{G^2-1}$$

$$\text{and } \tan\mu = \epsilon' = \frac{FQ+PG}{FP-GQ} = \frac{G(F+P)}{FP-G^2}, \quad \text{since } G = Q$$

$$\therefore \quad \frac{2G}{G^2-1} = \frac{G(F+P)}{FP-G^2}$$

$$\text{or} \quad (F+P)(G^2-1) = 2(FP-G^2)$$

which is only a simplified form of (9,5)

From the form (9,2) of Born's action function, we can immediately find the form of the Lagrangian. From (4,1)

$$L = \tfrac{1}{2}T + R = \tfrac{1}{2}T + T_\lambda$$

$$= \frac{\cosh\frac{1}{2}\lambda}{\cos\frac{1}{2}\mu} - 1 + \frac{\sinh\frac{1}{2}\lambda}{\cos\frac{1}{2}\mu} = \frac{e^{\lambda/2}}{\cos\frac{1}{2}\mu} - 1$$

From (9,3) and (9,4) this reduces to

(14) II., p.73, equation (4,1)

(15) See, for example, V., p.576; equation (8)

$$L = \sqrt{1+2F-G^2} - 1,$$

which is the standard form of Born's lagrangian.

An alternative form for T can also be obtained in terms of the complex invariants ϕ and ψ. Using (9,3)

$$\left.\begin{aligned} \phi(1+\cos\mu) &= e^{\lambda} - e^{i\mu} \\ \psi(1+\cos\mu) &= e^{-\lambda} - e^{-i\mu} \end{aligned}\right\}$$

By addition,

$$\cos^2\tfrac{1}{2}\mu(\phi+\psi) = \cosh\lambda - \cos\mu$$

or $\quad 1+\cosh\lambda = \cos^2\tfrac{1}{2}\mu\,(\phi+\psi+2)$

ie $\quad 2\phi+2\psi+4 = \dfrac{4\cosh^2\frac{1}{2}\lambda}{\cos^2\frac{1}{2}\mu}$ and (9,2) can be written as

$$T = \sqrt{2\phi+2\psi+4} - 2 \qquad (9,6)$$

From (9,6) it easily follows that $T =$ Lagrangian + Hamiltonian.

In fact,

$2\phi+2\psi+4 = 2F+2P+4$. Since $G = Q$ in Born's case, and from a known identity(16)

$$\sqrt{2F+2P+4} = L+H+2$$

The form (9,6) is more general than the one given by

(16) V., p.580., equation (23)

Weiss[17] which holds when the invariant G is neglected. The complex Lagrangian and Hamiltonian given there for this case are respectively $(1+2\phi)^{1/2}-1$, and $(1+2\psi)^{1/2}-1$, and in this case we have

$$\sqrt{1+2\phi}+\sqrt{1+2\psi}-2=\sqrt{2\phi+2\psi+4}-2 \qquad (9,7)$$

in virtue of the relation

$$\phi+2\phi\psi+\psi=0 \qquad (9,8)$$

while the left-hand side of (9,7) is <u>not</u> Born's action function in the general case, we have shown that the right hand side is so.

10. Other action functions.

The one parameter group of action functions given by Infeld, and the one given by Hoffmann and Infeld are given in our notation by

and following the analogy of Born's action-function we can use $\cos\frac{1}{2}\mu$ or $(1/\cos\frac{1}{2}\mu)$ which tends to unity with $\mu=0$, as a sort of guage-factor, to generalise (10,1). It appears, however, that there is no criterion which leads to such a kind of generalisation

(17) IV., p.91, equation (8,5).

Also $T_\lambda = R$ gives

$$e^{\lambda/2} = (1+2R)\cos\tfrac{1}{2}\mu \tag{10,4}$$

From (6,8) and (6,9) - using (10,3), (10,4)

$$\left.\begin{aligned} 2F &= (1+2R)^2 - (1+2R)\cos^2\tfrac{1}{2}\mu \\ 2G &= -(1+2R)\cos\tfrac{1}{2}\mu\,\sin\tfrac{1}{2}\mu \end{aligned}\right\} \tag{10,5}$$

Since our idea is to obtain the power-series expansions of the Lagrangian obtained from (10,2) up to the second power in F and G, we assume(19)

$$\left.\begin{aligned} R &= a_0 F + a_1 F^2 + a_2 G^2 \\ \cos\tfrac{1}{2}\mu &= 1 + b_0 F + b_1 F^2 + b_2 G^2 \end{aligned}\right\} \tag{10,6}$$

since $R \to 0$ as $(F,G) \to 0$ while $\cos\tfrac{1}{2}\mu \to 1$. Substituting (10,6) in (10,5) after squaring the second equation in the latter, equating coefficients of like powers and neglecting terms of order higher than two, we get

$$\left.\begin{aligned} a_0 = 1,\ a_1 = -2,\ b_1 = -2 \\ b_0 = b_1 = 0,\ b_2 = -2 \end{aligned}\right\}$$

i.e.

$$\left.\begin{aligned} R &= F - 2F^2 - 2G^2 \\ \cos\tfrac{1}{2}\mu &= 1 - 2G^2 \end{aligned}\right\} \tag{10,6a}$$

---

(19) The invariant G appears as square only, as pointed out by Weiss. See V., p.89.

The Lagrangian corresponding to (10,2) is given by

$$L = \frac{1}{2}T + R$$
$$= 2R - \frac{1}{2}\lambda, \qquad (\text{using } (10,4)) \qquad (10,7)$$

Again from (10,4) and (10,6a), we can write

$$\lambda = \log(1 + 4F - 4F^2 - 12G^2)$$

$$= 4F - 12F^2 - 12G^2, \quad (\text{up to second power})$$

and (10,7) reduces to

$$L = F - F^2 - G^2$$

or $$L = \frac{1}{2}(\vec{B}^2 - \vec{E}^2) - \frac{1}{4}\left\{(\vec{B}^2 - \vec{E}^2)^2 + 4(\vec{B}\vec{E})^2\right\} \qquad (10,8)$$

whereas the Lagrangian of Euler and Köckel is

$$L = \frac{1}{2}(\vec{B}^2 - \vec{E}^2) - \frac{1}{90\pi m^4\alpha}\left\{(\vec{B}^2 - \vec{E}^2)^2 + 7(\vec{B}\vec{E})^2\right\} \qquad (10,9)$$

and Born's Lagrangian is,

$$L = \frac{1}{2}(\vec{B}^2 - \vec{E}^2) - \frac{1}{8}\left\{(\vec{B}^2 - \vec{E}^2)^2 + 4(\vec{B}\vec{E})^2\right\} \qquad (10,10)$$

Thus while (10,8) and (10,10) agree in the second order terms in G, neither of them coincides with (10,9). Also (10,8) gives the same estimate $\frac{1}{m} \simeq 130$ as done by Infeld's action function $(\gamma = 0)$.

11. Coincidence with Lagrangian of Euler and Kockel.

The question that now arises is whether it is possible to manipulate the action-function so as to obtain this coincidence with (10,9) and so as to reduce to the Hoffmann-Infeld function when $\mu = 0$. Now it can be shown that if we alter the second term of (10,2) by multiplying or dividing it with the factor $\cos\frac{1}{2}\mu$, the relations between the invariants F, G, P, Q will not be algebraic. Since this is desirable on grounds of simplicity, we try with

$$T = \left(e^{\lambda/2}/\cos\tfrac{1}{2}\mu\right) - \tfrac{1}{2}\lambda - f\left(\cos\tfrac{1}{2}\mu\right) \quad \text{such that } f(1) = 1$$

$$\text{or } T = \left(e^{\lambda/2}/\cos\tfrac{1}{2}\mu\right) - \tfrac{1}{2}\lambda - f(\nu), \quad (\nu = \cos\tfrac{1}{2}\mu) \qquad (11,1)$$

$$R = T_\lambda = \left(e^{\lambda/2}/2\cos\tfrac{1}{2}\mu\right) - \tfrac{1}{2}\,; \text{ ie } e^{\lambda/2} = (1+2R)\cos\tfrac{1}{2}\mu \qquad (10,4)$$

$$T_\mu = \left(e^{\lambda/2}\sin\tfrac{1}{2}\mu\right)/2\cos^2\tfrac{1}{2}\mu + \tfrac{1}{2}f'(\nu)\sin\tfrac{1}{2}\mu$$

$$L = 2R - \tfrac{1}{4}\lambda + \tfrac{1}{2}\left\{1 - f(\nu)\right\} \qquad (11,2)$$

In consonance with (10,6a) we assume, up to the second order

$$\left.\begin{aligned} R &= F - aF^2 + \alpha G^2 \\ \cos\tfrac{1}{2}\mu &= 1 + \beta G^2 \\ f(\nu) &= 1 + l G^2 \\ f'(\nu) &= k + k' G^2 \end{aligned}\right\} \qquad (11,3)$$

From (6,8) - (6,9),

$$\left.\begin{aligned} 2F &= (1+2R)^2-(1+2R)\cos^2\tfrac{1}{2}\mu+(1+2R)f'(\nu)\sin^2\tfrac{1}{2}\mu\cos\tfrac{1}{2}\mu \\ -2G &= (1+2R)f'(\nu)\sin\tfrac{1}{2}\mu\cos^2\tfrac{1}{2}\mu+(1+2R)\sin\tfrac{1}{2}\mu\cos\tfrac{1}{2}\mu \end{aligned}\right\} \qquad (11,4)$$

Substituting (11,3) in (11,4) after squaring its second equation, retaining terms up to the second order, and comparing coefficients

$$\left.\begin{aligned} \beta &= -2/(k+1)^2 \\ \alpha &= -2/(k+1) \end{aligned}\right\} \qquad (11,5)$$

From (10,4) we find λ, and substitute for λ, R, and $f(\gamma)$ in (11,2) obtaining

$$L = F-2F^2+G^2\left(\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\ell\right) \qquad (11,6)$$

If (11,6) should coincide with (10,9) in the coefficient of G^2, we should have

$$\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\ell = -7/4$$

$$\text{or} \quad 2\beta+2\ell-4\alpha = 7 \qquad (11,7)$$

We now still further specialise (11,1) by putting $f(\nu)=\nu^n$, and determine the value of n, so as to satisfy (11,7).

$$f(\nu)=\nu^n=(1+\beta G^2)^n=1+n\beta G^2,$$
$$f'(\nu)=n\nu^{n-1}=n+n(n-1)\beta G^2.$$

Hence,
$$\left.\begin{aligned} l &= n\beta \\ K &= n \end{aligned}\right\} \qquad (11,8)$$

Putting (11,5) and (11,8) in (11,7), we get the equation for n,

$$\frac{8}{n+1} - \frac{4}{(n+1)^2} - \frac{4n}{(n+1)^2} = 7$$

Or,
$$7n^2 + 10n + 3 = 0 \qquad (11,9)$$

giving $n = -1$, or $n = -3/7$, but $n = -1$ would make both α and β infinite and consequently ~~of~~ R and $\cos\frac{1}{2}\mu$. Neglecting this value, we can take $n = -3/7$, and obtain an action-function which has the desired coincidence of the Lagrangian.

In an exactly similar way, we can show that

$$T = e^{\lambda/2}\cos\tfrac{1}{2}\mu - \tfrac{1}{2}\lambda - (\cos\tfrac{1}{2}\mu)^n, \text{ where } n = 8/7,$$

also gives an action-function of this type.

12. Conclusion.

It is practically certain that the method adopted above can be applied to several other functions so as to obtain the desired Lagrangian. It appears therefore that even this criterion does not restrict the type of function; nor should this be surprising since we are

merely comparing coefficients of particular terms in two power-series expansions. Anyway, the several types of action-functions serve to bring out the possibilities of the new field theory.

--- 0 ---

(7)

COMPLEX REPRESENTATION IN BORN'S FIELD THEORY.

Reprinted from "The Proceedings of the Indian Academy of Sciences",
Vol. IV, No. 5, Sec. A, 1936.

COMPLEX REPRESENTATION IN BORN'S FIELD THEORY.

BY B. S. MADHAVA RAO.
(University of Mysore.)
(From the Department of Physics, Indian Institute of Science, Bangalore.)

Received October 27, 1936.

1. *Introduction.*

SCHRODINGER[1] has given a representation of Born's field theory by using two complex combinations of **B**, **E**, **H**, **D**

$$\mathcal{F} = \mathbf{B} - i\mathbf{D}\,;\ \mathcal{G} = \mathbf{E} + i\mathbf{H}, \quad \ldots \quad (1)$$

starting with the Lagrangian

$$\mathcal{L} = \frac{\mathcal{F}^2 - \mathcal{G}^2}{(\mathcal{F}\mathcal{G})} \quad \ldots \quad (2)$$

and the " condition of conjugateness " given by

$$\left.\begin{aligned} \mathcal{F}^* &= \frac{\partial \mathcal{L}}{\partial \mathcal{G}} = -\frac{2\mathcal{G}}{(\mathcal{F}\mathcal{G})} - \frac{\mathcal{F}^2 - \mathcal{G}^2}{(\mathcal{F}\mathcal{G})^2}\,\mathcal{F} \\ \mathcal{G}^* &= \frac{\partial \mathcal{L}}{\partial \mathcal{F}} = \frac{2\mathcal{F}}{(\mathcal{F}\mathcal{G})} - \frac{\mathcal{F}^2 - \mathcal{G}^2}{(\mathcal{F}\mathcal{G})^2}\,\mathcal{G} \end{aligned}\right\} \quad \ldots \quad (3)$$

The equivalence of this treatment with Born's theory is shown by using suitable Lorentz and γ-transformations and reducing both representations to a common form. In this paper, I have established this equivalence directly by means of analytical transformations by showing that (3) is an exact transcription of Born's relations between primary and secondary field vectors and also given two other simple proofs of this equivalence. This has necessitated a detailed study of the invariants of Born's and Schrödinger's representations and as a result I have been able to find two other alternative complex representations (entirely equivalent to Schrödinger's) in which the action function again appears with the square root. The results obtained in this paper are summarised as follows :—

(1) If $\mathcal{G}^* = \frac{\partial \mathcal{L}}{\partial \mathcal{F}}$ and $\mathcal{F}^* = \frac{\partial \mathcal{L}}{\partial \mathcal{G}}$ and Born's relations hold between the real field vectors, then $\mathcal{L}$ has necessarily the form (2).

(2) A detailed study is made of the relations between the several invariants and space invariants of Born's theory.

[1] E. Schrödinger, *Proc. Roy. Soc.*, A. 1935, **150**, 465.

VIII

(3) A similar study is made of the invariants of Schrödinger's representation and its equivalence with Born's representation is exhibited analytically.

(4) It is shown that Schrödinger's representation is equivalent to the alternative representations

$$\left.\begin{aligned}\frac{\partial \mathcal{U}}{\partial \mathcal{F}} &= \mathcal{G}^* \\ \frac{\partial \mathcal{U}}{\partial \mathcal{F}^*} &= -\mathcal{G}\end{aligned}\right\} \text{ and } \left.\begin{aligned}\frac{\partial \mathcal{V}}{\partial \mathcal{G}} &= \mathcal{F}^* \\ \frac{\partial \mathcal{V}}{\partial \mathcal{G}^*} &= -\mathcal{F}\end{aligned}\right\}$$

where $\mathcal{U}$ and $\mathcal{V}$ are functions respectively of $\mathcal{F}$, $\mathcal{F}^*$ and $\mathcal{G}$, $\mathcal{G}^*$ with a square root form. These representations lead in the simplest manner to the form (2) of Schrödinger's Lagrangian.

2. *Form of Schrödinger's Lagrangian.*

We will show that if $\mathcal{G}^* = \frac{\partial \mathcal{L}}{\partial \mathcal{F}}$ and $\mathcal{F}^* = \frac{\partial \mathcal{L}}{\partial \mathcal{G}}$ and if Born's relations hold between the field components, then $\mathcal{L}$ has necessarily the form (2).

If $\mathcal{F}$ and $\mathcal{G}$ are defined by (1), they form a true six vector defined by the antisymmetric tensor

$$q_{kl} = f_{kl} - i p_{kl}^* \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4)$$

where the tensors f_{kl} and p_{kl} define the field components of Born's theory.[2] The complex conjugates[3] ($\mathcal{G}^*$, $\mathcal{F}^*$) also form a true six vector defined by the antisymmetric tensor

$$r^{kl} = f^{*kl} - i p^{kl} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (5)$$

and the equations

$$\mathcal{G}^* = \frac{\partial \mathcal{L}}{\partial \mathcal{F}}\,;\ \mathcal{F}^* = \frac{\partial \mathcal{L}}{\partial \mathcal{G}}$$

can be written in the form

$$r^{kl} = \frac{\partial \mathcal{L}}{\partial q_{kl}} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (6)$$

With $\mathcal{L}$ we can associate, just as in Born's theory, a Hamiltonian $\mathcal{H}$ given by

$$\mathcal{H} = \mathcal{L} - \tfrac{1}{2}\, r^{kl} q_{kl} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (7)$$

Following the notation of Born and Infeld we shall use the relations[4]

$$\frac{1 + \mathrm{F} - \mathrm{G}^2}{1 + \mathrm{G}^2} = \frac{1 + \mathrm{Q}^2}{1 + \mathrm{P} - \mathrm{Q}^2} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (8)$$

---

[2] *i.e.*, $(p_{23}, p_{31}, p_{12}) \rightarrow \mathbf{H}$; $(f_{23}, f_{31}, f_{12}) \rightarrow \mathbf{B}$
$(p_{14}, p_{24}, p_{34}) \rightarrow \mathbf{D}$; $(f_{14}, f_{24}, f_{34}) \rightarrow \mathbf{E}$

[3] The * indicates the dual when associated with a tensor and the complex conjugate when referring to a vector.

[4] Born and Infeld, *Proc. Roy. Soc.*, A, 1934, **144**, 435.

$$\mathrm{G} = \mathrm{Q} \qquad \ldots \qquad (9)$$

which hold between the several invariants. Calculating the second term on the right-hand side of (7), by using (1),

$$\tfrac{1}{2}\, r^{kl} q_{kl} = (\mathbf{E} - i\,\mathbf{H})\,(\mathbf{B} - i\,\mathbf{D}) + (\mathbf{B} + i\,\mathbf{D})\,(\mathbf{E} + i\,\mathbf{H})$$
$$= 2\,(\mathrm{G} - \mathrm{Q}) = 0, \text{ using (9)} \qquad \ldots \qquad (10)$$

Hence $\mathcal{H} = \mathcal{L}$ or, *in Schrödinger's representation the Hamiltonian coincides with the Lagrangian*; and corresponding to (6) we also have

$$q^{*kl} = \frac{\partial \mathcal{L}}{\partial r^{*}{}_{kl}} \qquad \ldots \qquad (11)$$

where $\mathcal{L}$ is defined as a function of the invariants associated with r^{kl}. In (6) let $\mathcal{L}$ be considered a function of the two invariants f and g, where

$$\left.\begin{aligned} f &= \mathcal{F}^2 - \mathcal{G}^2 = \tfrac{1}{2}\, q_{kl}\, r^{kl} \\ g &= (\mathcal{F}\,\mathcal{G}) = \tfrac{1}{4}\, q_{kl}\, q^{*kl} \end{aligned}\right\} \qquad \ldots \qquad (12)$$

and in (11) as a function of the two invariants p and q, where

$$\left.\begin{aligned} p &= \mathcal{G}^{*2} - \mathcal{F}^{*2} = \tfrac{1}{2}\, r_{kl}\, r^{kl} \\ q &= (\mathcal{G}^{*}\,\mathcal{F}^{*}) = -\tfrac{1}{4}\, r_{kl}\, r^{*kl} \end{aligned}\right\} \qquad \ldots \qquad (13)$$

(6) and (11) can be written in the form

$$r^{kl} = 2\,\frac{\partial \mathcal{L}}{\partial f}\, q^{kl} + \frac{\partial \mathcal{L}}{\partial g}\, q^{kl} \qquad \ldots \qquad (6, \mathrm{A})$$

$$-\, q^{*kl} = 2\,\frac{\partial \mathcal{L}}{\partial p}\, r^{*kl} + \frac{\partial \mathcal{L}}{\partial q}\, r^{kl} \qquad \ldots \qquad (11, \mathrm{A})$$

By direct calculation, using (4) and (5) we can easily deduce

$$\tfrac{1}{2}\, r^{kl} q_{kl}{}^{*} = \tfrac{1}{2}\, r^{*kl} q_{kl} = \tfrac{1}{2}\, r_{kl} q^{*kl} = \tfrac{1}{2}\, r_{kl}{}^{*} q^{kl} = -\,(\mathrm{F} + \mathrm{P}) \qquad \ldots \qquad (14)$$

We now multiply (6, A) respectively by $\tfrac{1}{4}\, q_{kl}$, $\tfrac{1}{4}\, r_{kl}{}^{*}$ and $\tfrac{1}{2}\, r_{kl}$ and sum up in each case. In view of (14), this gives the relations

$$\left.\begin{aligned} f\,\frac{\partial \mathcal{L}}{\partial f} + g\,\frac{\partial \mathcal{L}}{\partial g} &= 0 \\ \frac{\partial \mathcal{L}}{\partial f} &= \frac{q}{\mathrm{F} + \mathrm{P}} \\ \frac{\partial \mathcal{L}}{\partial g} &= -\,\frac{p}{\mathrm{F} + \mathrm{P}} \end{aligned}\right\} \qquad \ldots \qquad (15)$$

Treating (11, A) in exactly the same way by multiplying by $\tfrac{1}{4}\, r_{kl}{}^{*}$, $\tfrac{1}{4}\, q_{kl}$ and $\tfrac{1}{2}\, q_{kl}{}^{*}$ and summing up, we get

$$\left.\begin{aligned} p\,\frac{\partial \mathcal{L}}{\partial p} + q\,\frac{\partial \mathcal{L}}{\partial q} &= 0 \\ \frac{\partial \mathcal{L}}{\partial p} &= \frac{g}{\mathrm{F} + \mathrm{P}} \\ \frac{\partial \mathcal{L}}{\partial q} &= -\,\frac{f}{\mathrm{F} + \mathrm{P}} \end{aligned}\right\} \qquad \ldots \qquad (16)$$

From the first of the equations in (15) or (16) we see that $\mathcal{L}$ is a homogeneous function of degree zero in (f, g) or (p, q), which is also otherwise obvious from the relation (10) or its equivalent

$$(\mathcal{F}\, \mathcal{G}^*) + (\mathcal{F}^* \mathcal{G}) = 0.$$

Also from either (15) or (16) we get

$$\frac{f}{g} = \frac{p}{q} \qquad \ldots \qquad (17)$$

We now proceed to calculate actually the values of the partial derivatives of $\mathcal{L}$ by making use of Born's relation (8). If L and H be the Lagrangian and Hamiltonian of Born's theory

$$\left.\begin{aligned} L &= \sqrt{1 + F - G^2} - 1 \\ H &= \sqrt{1 + P - Q^2} - 1 \\ H &= L - R \end{aligned}\right\}$$

and using these expressions (8) can be written alternatively as

$$(L + 1)(H + 1) = 1 + G^2 \qquad \ldots \qquad (18)$$

From (1) we get

$$g = G + Q + iR = 2G + iR$$
$$q = G + Q - iR = 2G - iR,$$

hence,

$$\begin{aligned} gq &= 4G^2 + R^2 = 4G^2 + [(L + 1) - (H + 1)]^2 \\ &= 4G^2 + (L + 1)^2 + (H + 1)^2 - 2(1 + G^2), \quad \text{from (18)} \\ &= 4G^2 + (1 + F - G^2) + (1 + P - Q^2) - 2(1 + G^2) \end{aligned}$$

$$\therefore \quad gq = F + P \qquad \ldots \qquad (19)$$

Substituting (19) in the second equations of (15) and (16),

$$\frac{\partial \mathcal{L}}{\partial f} = \frac{1}{g}; \quad \frac{\partial \mathcal{L}}{\partial p} = \frac{1}{q}$$

and the first equations of the same two sets show that

$$\frac{\partial \mathcal{L}}{\partial g} = -\frac{f}{g^2}; \quad \frac{\partial \mathcal{L}}{\partial q} = -\frac{p}{q^2}$$

Hence,

$$\mathcal{L} = \frac{f}{g} = \frac{p}{q} \qquad \ldots \qquad (20)$$

which is the same as (2). Equation (20) consists of both the equations (1) and (6) of Schrödinger's paper.

3. *Relations between Born's Invariants.*

The invariants F, G, P, Q, R, S are defined by the relations

$$\begin{aligned} F &= \tfrac{1}{2} f^{kl} f_{kl} = -\tfrac{1}{2} f^{*kl} f^*_{kl}; & P &= \tfrac{1}{2} p_{kl}^* p^{*kl} = -\tfrac{1}{2} p_{kl} p^{kl}; \\ G &= \tfrac{1}{4} f_{kl} f^{*kl} = \tfrac{1}{4} f_{kl}^* f^{kl}; & Q &= \tfrac{1}{4} p^{kl} p_{kl}^* = \tfrac{1}{4} p^{*kl} p_{kl}; \\ R &= \tfrac{1}{2} f_{kl} p^{kl} = -\tfrac{1}{2} f^*_{kl} p^{*kl}; & S &= \tfrac{1}{2} f^{kl} p_{kl}^* = \tfrac{1}{2} f^{*kl} p_{kl}; \end{aligned}$$

or, in space vector notation,

$$F = \mathbf{B}^2 - \mathbf{E}^2 \; ; \; P = \mathbf{D}^2 - \mathbf{H}^2 \; ; \; G = (\mathbf{B}\,\mathbf{E}) \; ; \; Q = (\mathbf{D}\,\mathbf{H}) \; ;$$
$$R = (\mathbf{B}\,\mathbf{H}) - (\mathbf{D}\,\mathbf{E}) \; ; \; S = (\mathbf{B}\,\mathbf{D}) + (\mathbf{E}\,\mathbf{H}).$$

We shall derive the several relations that exist between these invariants.

In the equations

$$\left.\begin{aligned} p^{kl} &= 2\frac{\partial L}{\partial F} f^{kl} + \frac{\partial L}{\partial G} f^{*kl} \\ f^{*kl} &= 2\frac{\partial H}{\partial P} p^{*kl} + \frac{\partial H}{\partial Q} p^{kl} \end{aligned}\right| \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

multiply the first respectively by $\frac{1}{2} f_{kl}$, $\frac{1}{2} f_{kl}^{*}$, $\frac{1}{2} p_{kl}$, $\frac{1}{4}$ p_{kl}^{*} and sum up; similarly the second by $\frac{1}{4} f_{kl}$, $\frac{1}{2} f_{kl}^{*}$, $\frac{1}{2} p_{kl}$, $\frac{1}{2} p_{kl}^{*}$ and sum up. We then get

$$\begin{array}{rl|rl} R = & 2F\dfrac{\partial L}{\partial F} + 2G\dfrac{\partial L}{\partial G} & G = & S\dfrac{\partial H}{\partial P} + \dfrac{R}{2}\dfrac{\partial H}{\partial Q} \\ S = & 4G\dfrac{\partial L}{\partial F} - F\dfrac{\partial L}{\partial G} & -F = & -2R\dfrac{\partial H}{\partial P} + S\dfrac{\partial H}{\partial Q} \\ -P = & 2R\dfrac{\partial L}{\partial F} + S\dfrac{\partial L}{\partial G} & S = & 4Q\dfrac{\partial H}{\partial P} - P\dfrac{\partial H}{\partial Q} \\ Q = & S\dfrac{\partial L}{\partial F} - \dfrac{R}{2}\dfrac{\partial L}{\partial G} & -R = & 2P\dfrac{\partial H}{\partial P} + 2Q\dfrac{\partial H}{\partial Q} \end{array}$$

Using $L = \sqrt{1 + F - G^2} - 1$; $H = \sqrt{1 + P - Q^2} - 1$, introducing the values of $\partial L/\partial F$, $\partial L/\partial G$, $\partial H/\partial P$, $\partial H/\partial Q$ and rearranging

$$\left.\begin{array}{lrl|rll} (a) & R = & \dfrac{F - 2G^2}{L + 1} & -R = & \dfrac{P - 2Q^2}{H + 1} & (a') \\ (b) & S = & \dfrac{G(F + 2)}{L + 1} & S = & \dfrac{Q(P + 2)}{H + 1} & (b') \\ (c) & -P = & \dfrac{R - GS}{L + 1} & F = & \dfrac{R + QS}{H + 1} & (c') \\ (d) & 2Q = & \dfrac{S + RG}{L + 1} & 2G = & \dfrac{S - RQ}{H + 1} & (d') \end{array}\right| \quad \ldots \quad (22)$$

All possible relations that can exist between the invariants can be derived out of the equations (22). We shall deduce a few important ones which will be of use later on.

Substituting (*a*) and (*b*) in (*d*) or (*a′*) and (*b′*) in (*d′*), we get

$$G = Q \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Substitution of (*a*) and (*b*) in (*c′*) or (*a′*) and (*b′*) in (*c*) gives

$$(L + 1)(H + 1) = 1 + G^2 \quad \ldots \quad (18)$$

Comparing (*b*) and (*b′*)

$$\frac{F + 2}{L + 1} = \frac{P + 2}{H + 1} = \frac{S}{G}$$

$$\therefore \quad \frac{S}{G} = \frac{F - P}{R} = \frac{F + P + 4}{L + H + 2}$$

Comparing (a) and (a')

$$R = \frac{F - 2\,G^2}{L + 1} = \frac{2\,Q^2 - P}{H + 1}$$

$$\therefore \quad R = \frac{F - P}{L + H + 2} = \frac{F + P - 4\,G^2}{R}$$

and we have the relations

$$\frac{S}{G} = \frac{F - P}{R} = L + H + 2 = \sqrt{F + P + 4} \qquad \ldots \qquad \ldots \qquad (23)$$

$$R^2 + 4\,G^2 = \frac{S^2}{G^2} - 4 = F + P \qquad \ldots \qquad \ldots \qquad \ldots \qquad (24)$$

Finally we can write (22) (d) and (d') in the form

$$\left.\begin{aligned} \frac{S}{G} + R &= 2\,(L + 1) \\ \frac{S}{G} - R &= 2\,(H + 1) \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (25)$$

We shall next deduce some relations existing between the several *space invariants* according to Born's theory. Let

$$F_1 = \mathbf{B}^2 + \mathbf{E}^2\,;\; P_1 = \mathbf{D}^2 + \mathbf{H}^2\,;\; M = (\mathbf{D}\,\mathbf{B})\,;\; N = (\mathbf{E}\,\mathbf{H})\,;$$
$$J = (\mathbf{D}\,\mathbf{E})\,;\; K = (\mathbf{B}\,\mathbf{H})$$
$$l_1 = (1 + \mathbf{B}^2)\,;\; m_1 = (1 + \mathbf{D}^2)\,;\; l_2 = (1 - \mathbf{H}^2)\,;\; m_2 = (1 - \mathbf{E}^2)$$
$$R_1 = (\mathbf{B}\,\mathbf{H}) + (\mathbf{D}\,\mathbf{E}) = K + J$$
$$S_1 = (\mathbf{B}\,\mathbf{D}) - (\mathbf{E}\,\mathbf{H}) = M - N$$

The action functions U and V of Born's theory can be written

$$U = \sqrt{l_1\,m_1 - M^2} - 1 = \sqrt{1 + \mathbf{B}^2 + \mathbf{D}^2 + \mathbf{S}^2} - 1$$
$$V = \sqrt{l_2\,m_2 - N^2} - 1 = \sqrt{1 - \mathbf{E}^2 - \mathbf{H}^2 + \mathbf{S}^2} - 1$$

where

$$\mathbf{S} = (\mathbf{D} \times \mathbf{B}) = (\mathbf{E} \times \mathbf{H})\ ^{5}$$

From the relations[6]

$$\left.\begin{aligned} U + 1 &= M/G \\ V + 1 &= N/G \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (26)$$

we have,

$$U + V + 2 = \frac{M + N}{G} = \frac{S}{G} = L + H + 2 \qquad \ldots \qquad \ldots \qquad (27)$$

Observing that $U = L + J$, and $V = L - K$, we have

$$2\,(U + 1) = 2\,(L + 1) + 2\,J = \frac{S}{G} + R + 2\,J, \qquad \text{from (25)}$$

$$= \frac{S}{G} + K - J + 2\,J = \frac{S}{G} + R_1$$

5 Born-Infeld, *Proc. Roy. Soc.*, A, 1935, **150**, 159.

6 Born-Infeld, *Proc. Roy. Soc.*, A, 1934, **147**, 545, equation (h).

hence the relations

$$\left.\begin{aligned} \frac{S}{G} + R_1 &= 2\ (U + 1) \\ \frac{S}{G} - R_1 &= 2\ (V + 1) \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

analogous to (25).

Again, $S_1 = M - N = G\ (U - V)$, from (26)

$$= G\ (L + J - L + K)$$

$$\therefore \quad S_1 = GR_1 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

The relations between the primary and secondary field vectors in Born's theory when L, H, and U, V are taken as the action functions, *viz.*,

$$\left.\begin{aligned} \mathbf{H} = \frac{\partial L}{\partial \mathbf{B}}; \ \mathbf{D} = -\frac{\partial L}{\partial \mathbf{E}} \\ \mathbf{B} = \frac{\partial H}{\partial \mathbf{H}}; \ \mathbf{E} = \frac{\partial H}{\partial \mathbf{D}} \end{aligned}\right\}$$

and,

$$\left.\begin{aligned} \mathbf{E} = \frac{\partial U}{\partial \mathbf{D}}; \ \mathbf{H} = \frac{\partial U}{\partial \mathbf{B}} \\ \mathbf{D} = -\frac{\partial V}{\partial \mathbf{E}}; \ \mathbf{B} = -\frac{\partial V}{\partial \mathbf{H}} \end{aligned}\right\}$$

can be written in the forms

$$\left.\begin{aligned} \mathbf{H}\ (L + 1) &= \mathbf{B} - G\ \mathbf{E} \\ \mathbf{D}\ (L + 1) &= \mathbf{E} + G\ \mathbf{B} \\ \mathbf{B}\ (H + 1) &= \mathbf{H} + G\ \mathbf{D} \\ \mathbf{E}\ (H + 1) &= \mathbf{D} - G\ \mathbf{H} \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$ [7]

and,

$$\left.\begin{aligned} \mathbf{E}\ (U + 1) &= l_1\ \mathbf{D} - M\ \mathbf{B} &&= \mathbf{D} + (\mathbf{B} \times \mathbf{S}) \\ \mathbf{H}\ (U + 1) &= m_1\ \mathbf{B} - M\ \mathbf{D} &&= \mathbf{B} - (\mathbf{D} \times \mathbf{S}) \\ \mathbf{D}\ (V + 1) &= l_2\ \mathbf{E} + N\ \mathbf{H} &&= \mathbf{E} - (\mathbf{H} \times \mathbf{S}) \\ \mathbf{B}\ (V + 1) &= m_2\ \mathbf{H} + N\ \mathbf{E} &&= \mathbf{H} + (\mathbf{E} \times \mathbf{S}) \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (31)$$

Substituting for M and N in (31) from (26) and using (30) we get easily

$$\left.\begin{aligned} l_1 &= (L + 1)\ (U + 1) \\ m_1 &= (H + 1)\ (U + 1) \\ l_2 &= (H + 1)\ (V + 1) \\ m_2 &= (L + 1)\ (V + 1) \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

---

[7] There is a misprint in Born-Infeld, *Proc. Roy. Soc.*, A, 1934, **144**, 438, second formula in (3, 10A) where in the numerator $\mathbf{D} + Q\,\mathbf{H}$ should read $\mathbf{D} - Q\,\mathbf{H}$

Multiplying the equations (31) scalarly by **D**, **B**, **E** and **H** respectively, we have

$$\left.\begin{aligned} J\ (U+1) &= \mathbf{D}^2 + \mathbf{S}^2 \\ K\ (U+1) &= \mathbf{B}^2 + \mathbf{S}^2 \\ J\ (V+1) &= \mathbf{E}^2 - \mathbf{S}^2 \\ K\ (V+1) &= \mathbf{H}^2 - \mathbf{S}^2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

Similarly multiplying (30) scalarly by **B**, **E**, **H**, **D** respectively

$$\left.\begin{aligned} K\ (L+1) &= \mathbf{B}^2 - G^2 \\ J\ (L+1) &= \mathbf{E}^2 + G^2 \\ K\ (H+1) &= \mathbf{H}^2 + G^2 \\ J\ (H+1) &= \mathbf{D}^2 - G^2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

From (34) we deduce immediately

$$\left.\begin{aligned} F_1 &= (L+1)\ R_1 \\ P_1 &= (H+1)\ R_1 \\ F_1 - P_1 &= R R_1 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

From (33) and (34), using $U = L + J = H + K$, and $V = L - K = H - J$ we have

$$K\ J = G^2 + \mathbf{S}^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

$$\mathbf{S}^2 = (\mathbf{B}^2\ \mathbf{D}^2) - (\mathbf{B}\ \mathbf{D})^2 = \left(\frac{K + G\ M}{H+1}\right)\left(\frac{J + G\ M}{L+1}\right) - M^2$$

$$= \frac{(K + G\ M)\ \ (J + G\ M)}{1 + G^2} - M^2 \text{ from } (18)$$

$$\therefore\ \mathbf{S}^2 = \frac{K\ J - M\ N}{1 + G^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

From (36), (37) and (26), we easily derive

$$(U+1)\ (V+1) = 1 - \mathbf{S}^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

Coming now to vector products of the field quantities, in addition to **S**, the two expressions $(\mathbf{B} \times \mathbf{H})$ and $(\mathbf{E} \times \mathbf{D})$ are equal (see Reference[5]). By suitably multiplying equations (30) and (31) vectorially with the proper field components we get at once

$$(\mathbf{B} \times \mathbf{H}) = (\mathbf{E} \times \mathbf{D}) = G\ \mathbf{S} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

In an entirely analogous manner we can derive from (30),

$$\left.\begin{aligned} (\mathbf{B} \times \mathbf{E}) &= -\ (L+1)\ \mathbf{S} \\ (\mathbf{H} \times \mathbf{D}) &= -\ (H+1)\ \mathbf{S} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

4. *Invariants of Schrodinger's Representation.*

Denoting the invariants of Schrödinger's representation corresponding to F, G, P, Q, R, S, by the small letters f, g, p, q, r, s we can derive relations between them by proceeding with (6, A) and (11, A) just as we did with

equations (21). For this purpose we need only add to equations (15) and (16) the relations obtained by multiplying (6, A) and (11, A) respectively by $\frac{1}{2} q_{kl}^*$ and $\frac{1}{2} r_{kl}$ and summing up. This gives

$$s = 4g\frac{\partial \mathcal{L}}{\partial f} - f\frac{\partial \mathcal{L}}{\partial g} \qquad \text{(15, A)}$$

$$s = 4q\frac{\partial \mathcal{L}}{\partial p} - p\frac{\partial \mathcal{L}}{\partial q} \qquad \text{(16, A)}$$

It is easily seen that the relations in (15), (16), (15, A), (16, A) can be reduced to the following independent relations

$$\left.\begin{aligned} r = 0\,;\ s = -gq = 4 + \frac{f^2}{g^2}, \\ \frac{f}{g} = \frac{p}{q} \end{aligned}\right\} \qquad (41)$$

when we substitute the proper values for the partial derivatives of $\mathcal{L}$.

The connection between Schrödinger's and Born's invariants can be written down at once from (1), (10) and (14),

$$\left.\begin{aligned} f &= \mathrm{F} - \mathrm{P} - 2\,i\mathrm{S} \\ g &= 2\mathrm{G} + i\mathrm{R} \\ p &= \mathrm{P} - \mathrm{F} - 2\,i\mathrm{S} \\ q &= 2\mathrm{G} + i\mathrm{R} \\ r &= 0 \\ s &= -(\mathrm{F} + \mathrm{P}) \end{aligned}\right\} \qquad (42)$$

and conversely,

$$\left.\begin{aligned} 4\mathrm{F} &= f - p - 2s \\ 4\mathrm{P} &= p - f - 2s \\ 4\mathrm{G} = 4\mathrm{Q} &= g + q \\ 2\mathrm{R} &= i\,(q - g) \\ 4\mathrm{S} &= i\,(f + p) \end{aligned}\right\} \qquad (43)$$

From (41), $s = 4 + \frac{f^2}{g^2}$ gives

$$-\frac{f^2}{g^2} = \mathrm{F} + \mathrm{P} + 4$$

$$= \frac{\mathrm{S}^2}{\mathrm{G}^2} \text{ from (23)}$$

$$\therefore \quad i\mathcal{L} = i\frac{f}{g} = \frac{\mathrm{S}}{\mathrm{G}} = \mathrm{L} + \mathrm{H} + 2 = \frac{\mathrm{F} - \mathrm{P}}{\mathrm{R}} \qquad (44)^{8}$$

giving an expression of Schrödinger's Lagrangian in terms of Born's invariants. An alternative derivation of this relation, which is perhaps simpler, is

[8] This relation has also been deduced by Marie A. Baudot, *Comptes Rendus*, 1936, 202, 909–21 and 1158–59.

obtained by writing

$$f = \mathrm{F} - \mathrm{P} - 2\,i\,\mathrm{S} = \frac{\mathrm{R\,S}}{\mathrm{G}} - 2\,i\,\mathrm{S}, \text{ from (23)}$$

$$= -\frac{i\,\mathrm{S}}{\mathrm{G}}(2\,\mathrm{G} + i\,\mathrm{R})$$

$$\therefore \quad \frac{f}{g} = -\,i\,\frac{\mathrm{S}}{\mathrm{G}} \text{ putting in the value of } g \text{ from (42)}$$

$$\textit{i.e.,} \quad i\mathcal{L} = \frac{\mathrm{S}}{\mathrm{G}}.$$

We shall *establish directly the analytical equivalence of the two representations* by showing that the equations (3) are an exact transcription of Born's relations (30) between the primary and secondary field vectors. We have therefore to show directly that

$$\left.\begin{aligned} \mathbf{B} + i\,\mathbf{D} &= -\frac{2}{g}(\mathbf{E} + i\,\mathbf{H}) - \frac{f}{g^2}(\mathbf{B} - i\,\mathbf{D}) \\ \mathbf{E} - i\,\mathbf{H} &= \frac{2}{g}(\mathbf{B} - i\,\mathbf{D}) - \frac{f}{g^2}(\mathbf{E} + i\,\mathbf{H}) \end{aligned}\right\}$$

or, using, $g = 2\mathrm{G} + i\,\mathrm{R}$ and $\dfrac{f}{g} = -\,i\,\dfrac{\mathrm{S}}{\mathrm{G}}$, that

$$\left.\begin{aligned} -2(\mathbf{E} + i\,\mathbf{H}) + i\,\frac{\mathrm{S}}{\mathrm{G}}(\mathbf{B} - i\,\mathbf{D}) &= (2\,\mathrm{G} + i\,\mathrm{R})(\mathbf{B} + i\,\mathbf{D}) \\ 2(\mathbf{B} - i\,\mathbf{D}) + i\,\frac{\mathrm{S}}{\mathrm{G}}(\mathbf{E} + i\,\mathbf{H}) &= (2\,\mathrm{G} + i\,\mathrm{R})(\mathbf{E} - i\,\mathbf{H}) \end{aligned}\right\}$$

Equating real and imaginary parts, this requires proving that

$$\left.\begin{aligned} \frac{\mathrm{S}}{\mathrm{G}}\mathbf{D} - 2\,\mathbf{E} &= 2\,\mathrm{G}\,\mathbf{B} - \mathrm{R}\,\mathbf{D} \\ \frac{\mathrm{S}}{\mathrm{G}}\mathbf{B} - 2\,\mathbf{H} &= \mathrm{R}\,\mathbf{B} + 2\,\mathrm{G}\,\mathbf{D} \end{aligned}\right\}$$

and,

$$\left.\begin{aligned} -\frac{\mathrm{S}}{\mathrm{G}}\mathbf{H} + 2\,\mathbf{B} &= 2\,\mathrm{G}\,\mathbf{E} + \mathrm{R}\,\mathbf{H} \\ \frac{\mathrm{S}}{\mathrm{G}}\mathbf{E} - 2\,\mathbf{D} &= \mathrm{R}\,\mathbf{E} - 2\,\mathrm{G}\,\mathbf{H} \end{aligned}\right\}$$

i.e.,

$$\left.\begin{aligned} \mathbf{D}\left(\frac{\mathrm{S}}{\mathrm{G}} + \mathrm{R}\right) &= 2\left(\mathbf{E} + \mathrm{G}\,\mathbf{B}\right) \\ \mathbf{H}\left(\frac{\mathrm{S}}{\mathrm{G}} + \mathrm{R}\right) &= 2\left(\mathbf{B} - \mathrm{G}\,\mathbf{E}\right) \\ \mathbf{B}\left(\frac{\mathrm{S}}{\mathrm{G}} - \mathrm{R}\right) &= 2\left(\mathbf{H} + \mathrm{Q}\,\mathbf{D}\right) \\ \mathbf{E}\left(\frac{\mathrm{S}}{\mathrm{G}} - \mathrm{R}\right) &= 2\left(\mathbf{D} - \mathrm{Q}\,\mathbf{H}\right) \end{aligned}\right\}$$

120

and in view of (25) these relations become identical with (30), thus establishing the equivalence.

Coming now to the space invariants of Schrödinger's representation, we shall calculate directly from (1) the several scalar and vector products of the complex field vectors.

$$\mathcal{F}^2 = \mathbf{B}^2 - \mathbf{D}^2 - 2\,i\,\mathrm{M} = \mathrm{R}\,(\mathrm{U}+1) - 2\,i\,\mathrm{G}\,(\mathrm{U}+1), \text{ from (33)}$$

and (26)

$$\left.\begin{aligned} &\text{i.e., } & \mathcal{F}^2 &= -\,i\,(\mathrm{U}+1)\,g \\ &\text{Similarly } & \mathcal{F}^{*2} &= i\,(\mathrm{U}+1)\,g \\ && \mathcal{G}^2 &= i\,(\mathrm{V}+1)\,g \\ && \mathcal{G}^{*2} &= -\,i\,(\mathrm{V}+1)\,g \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (45)$$

$$\mathcal{F}^2 + \mathcal{G}^2 = \mathrm{F}_1 - \mathrm{P}_1 - 2i\mathrm{S}_1 = \mathrm{RR}_1 - 2i\mathrm{GR}_1, \text{ from (35) and (29)}$$

$$\left.\begin{aligned} &\text{i.e., } \mathcal{F}^2 + \mathcal{G}^2 = -\,i\,\mathrm{R}_1\,g \\ &f = \mathcal{F}^2 - \mathcal{G}^2 = -\,i\,\frac{\mathrm{S}}{\mathrm{G}}\,g \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (46)$$

Regarding the scalar products of different vectors,

$$\left.\begin{aligned} (\mathcal{F}\mathcal{G}) &= 2\,\mathrm{G} + i\,\mathrm{R} = g \\ (\mathcal{F}^*\mathcal{G}^*) &= 2\,\mathrm{G} - i\,\mathrm{R} = q \\ (\mathcal{F}\mathcal{G}^*) &= -\,i\,\mathrm{R}_1 \\ (\mathcal{F}^*\mathcal{G}) &= i\,\mathrm{R}_1 \\ (\mathcal{F}\mathcal{F}^*) &= \mathbf{B}^2 + \mathbf{D}^2 \\ (\mathcal{G}\mathcal{G}^*) &= \mathbf{E}^2 + \mathbf{H}^2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (47)$$

From (46) and (47) we immediately observe that

$$\mathcal{F}^2 + \mathcal{G}^2 = (\mathcal{F}\mathcal{G}^*)\,(\mathcal{F}\mathcal{G}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (48)$$

Finally, calculating the vector products,

$$\begin{aligned} \mathcal{F} \times \mathcal{G} &= \{(\mathbf{B} \times \mathbf{E}) + (\mathbf{D} \times \mathbf{H})\} + i\,\{(\mathbf{B} \times \mathbf{H}) - (\mathbf{D} \times \mathbf{E})\} \\ &= -\,\mathrm{R}\,\mathbf{S} + 2\,i\,\mathrm{G}\,\mathbf{S}, \text{ using (39) and (40)} \\ &= i\,g\,\mathbf{S} \end{aligned}$$

Also,

$$(\mathcal{F} \times \mathcal{G}^*) = (\mathbf{B} \times \mathbf{E}) - (\mathbf{D} \times \mathbf{H}) = -\,\mathbf{S}\,(\mathrm{L} + \mathrm{H} + 2) = -\,\frac{\mathrm{S}}{\mathrm{G}}\,\mathbf{S}$$

and $(\mathcal{F} \times \mathcal{F}^*) = -\,2\,i\,\mathbf{S}$, and similarly for other vector products. Collecting these together,

$$\left.\begin{aligned} (\mathcal{F} \times \mathcal{G}) &= i\,g\,\mathbf{S}; \quad (\mathcal{F}^* \times \mathcal{G}^*) = -\,i\,q\,\mathbf{S} \\ (\mathcal{F} \times \mathcal{G}^*) &= -\,\frac{\mathrm{S}}{\mathrm{G}}\,\mathbf{S}; \quad (\mathcal{F}^* \times \mathcal{G}) = -\,\frac{\mathrm{S}}{\mathrm{G}}\,\mathbf{S} \\ (\mathcal{F} \times \mathcal{F}^*) &= -\,2\,i\,\mathbf{S}; \quad (\mathcal{G} \times \mathcal{G}^*) = -\,2\,i\,\mathbf{S} \end{aligned}\right\} \quad \ldots \quad (49)$$

From equations (49),

$$(\mathcal{F}^* \times \mathcal{F}) = (\mathcal{G}^* \times \mathcal{G}) = \frac{2}{(\mathcal{F}\,\mathcal{G})}(\mathcal{F} \times \mathcal{G}) \quad .. \quad .. \quad (50)$$

which is equation (13) of Schrödinger's paper. It might also be noticed that all the vector products in (49) are expressed in terms of **S** which becomes equal to zero when $(\mathcal{F} \times \mathcal{G}) = 0$ and $g \neq 0$, so that all the vector products reduce to zero. This corresponds to the Lorentz-transformation which reduces the Maxwell-tensor to the diagonal and makes all the four composing three vectors parallel.

5. *Alternative Complex Representations.*

No complex combinations of the field vectors other than (1) are possible in view of the form of the Maxwell-Born field equations, *viz.*,

$$\left.\begin{aligned} \text{rot } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,;\ \text{rot } \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0 \\ \text{div } \mathbf{B} = 0\,;\ \text{div } \mathbf{D} = 0 \end{aligned}\right\}$$

but corresponding to the representations in Born's theory in which the action function is taken as the displacement energy density U or the field energy density V, we can set up analogous complex representations. Corresponding to the equation

$$\mathrm{U} = \mathrm{L} + (\mathbf{D}\,\mathbf{E})$$

of Born's theory, we introduce here

$$\mathcal{U} = \mathcal{L} + (\mathcal{F}\,\mathcal{G}^*) = \mathcal{L} - (\mathcal{F}^*\,\mathcal{G}) \quad .. \quad .. \quad .. \quad (51)$$

or, using (48)

$$\mathcal{U} = \mathcal{L} + \frac{\mathcal{F}^2 + \mathcal{G}^2}{(\mathcal{F}\,\mathcal{G})} = \frac{2\,\mathcal{F}^2}{(\mathcal{F}\,\mathcal{G})} \quad .. \quad .. \quad .. \quad (52)$$

From (52) it is seen that the $\mathcal{U}$ we have introduced is twice the component T_{44} of Schrödinger's energy-impulse tensor. If we now express $\mathcal{U}$ as a function of $\mathcal{F}$, $\mathcal{F}^*$ and treat them as primary field quantities, (51) enables us to find $\mathcal{G}$, $\mathcal{G}^*$ as derivaitves of $\mathcal{U}$ with respect to these. In fact

$$\begin{aligned} d\mathcal{U} &= d\mathcal{L} - \mathcal{F}^*\,d\mathcal{G} - \mathcal{G}\,d\mathcal{F}^* \\ &= \frac{\partial \mathcal{L}}{\partial \mathcal{F}}\,d\mathcal{F} + \frac{\partial \mathcal{L}}{\partial \mathcal{G}}\,d\mathcal{G} - \mathcal{F}^*\,d\mathcal{G} - \mathcal{G}\,d\mathcal{F}^* \\ &= \mathcal{G}^*\,d\mathcal{F} + \mathcal{F}^*\,d\mathcal{G} - \mathcal{F}^*\,d\mathcal{G} - \mathcal{G}\,d\mathcal{F}^* \\ &= \mathcal{G}^*\,d\mathcal{F} - \mathcal{G}\,d\mathcal{F}^* \end{aligned}$$

and hence

$$\left.\begin{aligned} \frac{\partial \mathcal{U}}{\partial \mathcal{F}} &= \mathcal{G}^* \\ \frac{\partial \mathcal{U}}{\partial \mathcal{F}^*} &= -\,\mathcal{G} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (53)$$

We shall now express $\mathcal{U}$ as a function of $\mathcal{F}$ and $\mathcal{F}^*$. Multiplying the first equation of (3) scalarly by $\mathcal{F}$, we get

$$\mathcal{F}\mathcal{F}^* = -2 - \frac{\mathcal{F}^2 - \mathcal{G}^2}{(\mathcal{F}\mathcal{G})^2}\mathcal{F}^2$$

$$= -2 - \frac{\mathcal{F}^4}{(\mathcal{F}\mathcal{G})^2} + \frac{\mathcal{F}^2\mathcal{G}^2}{(\mathcal{F}\mathcal{G})^2}$$

$$= -2 - \frac{\mathcal{F}^4}{(\mathcal{F}\mathcal{G})^2} + \frac{(\mathcal{F}\times\mathcal{G})^2 + (\mathcal{F}\mathcal{G})^2}{(\mathcal{F}\mathcal{G})^2}$$

$$= -1 - \left(\frac{\mathcal{F}^2}{\mathcal{F}\mathcal{G}}\right)^2 + \frac{(\mathcal{F}\times\mathcal{G})^2}{(\mathcal{F}\mathcal{G})^2}$$

Using (52) and (50), this can be written as,

$$\mathcal{F}\mathcal{F}^* = -1 - \tfrac{1}{4}\mathcal{U}^2 + \tfrac{1}{4}(\mathcal{F}\times\mathcal{F}^{2*})$$

or,

$$\mathcal{U} = -2i\sqrt{1 + \mathcal{F}\mathcal{F}^* - \tfrac{1}{4}(\mathcal{F}\times\mathcal{F}^*)^2} \qquad \ldots \qquad \ldots \quad (54)$$

Putting in the values of $(\mathcal{F}\mathcal{F}^*)$ and $(\mathcal{F}\times\mathcal{F}^*)$ from (47) and (49) in the square root on the right-hand side of (54), we get the significant result

$$\mathcal{U} = -2i(\mathrm{U}+1) = -2i\sqrt{1 + \mathbf{B}^2 + \mathbf{D}^2 + \mathbf{S}^2} \qquad \ldots \quad (55)$$

We could also have deduced (55) immediately from (52) by substituting in the latter the value of $\mathcal{F}^2$ from (45). Equation (54) or (55) appears to contain the essence of the reason why a complex representation of Born's field theory is possible. Of the four actions L, H, U, V possible in Born's theory we see from (54) and (55) that it is only in the last two cases that we can express the action functions directly as functions of the complex combinations specified by (1). This fact also provides the simplest proof of the equivalence of Born's and Schrödinger's representations. For, with

$$\mathcal{U} = -2i(\mathrm{U}+1)$$

we can directly establish (as we shall do a little farther) that the equations (53) are satisfied. We can now put

$$\mathcal{L} = \mathcal{U} - (\mathcal{F}\mathcal{G}^*) = \mathcal{U} + (\mathcal{F}^*\mathcal{G})$$

and show that equations (3) are satisfied. For the value of $\mathcal{L}$ as a function of $\mathcal{F}$ and $\mathcal{G}$,

$$\mathcal{L} = \mathcal{U} - (\mathcal{F}\mathcal{G}^*) \text{ gives}$$

$$\mathcal{L} = \frac{2\mathcal{F}^2}{(\mathcal{F}\mathcal{G})} - \frac{\mathcal{F}^2 + \mathcal{G}^2}{(\mathcal{F}\mathcal{G})}, \text{ from (52) and (48)}$$

$$\text{or } \mathcal{L} = \frac{\mathcal{F}^2 - \mathcal{G}^2}{(\mathcal{F}\mathcal{G})} \qquad (2)$$

which is Schrödinger's form of the Lagrangian.

Just as equations (3) are a transcription of Born's relations (30), we will show directly that the equations (53) are completely equivalent to (31)

Differentiating (54) with respect to $\mathscr{F}$ and $\mathscr{F}^*$

$$(U + 1) \frac{\partial \mathscr{U}}{\partial \mathscr{F}} = -i \left\{\mathscr{F}^* - \tfrac{1}{2} [\mathscr{F}^* \times (\mathscr{F} \times \mathscr{F}^*)]\right\} \quad \text{..(53, A)}$$

$$-(U + 1) \frac{\partial \mathscr{U}}{\partial \mathscr{F}^*} = i \left\{\mathscr{F} + \tfrac{1}{2} [\mathscr{F} \times (\mathscr{F} \times \mathscr{F}^*)]\right\} \quad \text{..(53, B)}$$

The right-hand side of (53, A) is equal to

$$\begin{aligned} & -i\{\mathscr{F}^* + i (\mathscr{F}^* \times \mathbf{S})\}, \text{ using (49)} \\ &= -i\{(\mathbf{B} + i\mathbf{D}) + i[(\mathbf{B} + i\mathbf{D}) \times \mathbf{S}]\} \\ &= \{\mathbf{D} + (\mathbf{B} \times \mathbf{S})\} - i\{\mathbf{B} - (\mathbf{D} \times \mathbf{S})\} \\ &= (\mathbf{E} - i\mathbf{H})(U + 1), \text{ using Born's relations (31)} \end{aligned}$$

Hence (53, A) reduces to

$$\frac{\partial \mathscr{U}}{\partial \mathscr{F}} = \mathscr{G}^*$$

Again, the right-hand side of (53, B) is equal to

$$\begin{aligned} & i\{\mathscr{F} - i(\mathscr{F} \times \mathbf{S})\} \\ &= i\{(\mathbf{B} - i\mathbf{D}) - i[(\mathbf{B} - i\mathbf{D}) \times \mathbf{S}]\} \\ &= \{\mathbf{D} + (\mathbf{B} \times \mathbf{S})\} + i\{\mathbf{B} - (\mathbf{D} \times \mathbf{S})\} \\ &= (\mathbf{E} + i\mathbf{H})(U + 1); \text{ and (53, B) reduces to} \end{aligned}$$

$$-\frac{\partial \mathscr{U}}{\partial \mathscr{F}^*} = \mathscr{G}$$

We can, next, introduce analogously the $\mathscr{V}$-representation by putting

$$\mathscr{V} = -2i\sqrt{1 - \mathscr{G}\mathscr{G}^* - \tfrac{1}{4}(\mathscr{G} \times \mathscr{G}^*)^2} \quad \text{..} \quad \text{..(54, A)}$$

$$\text{or} \quad \mathscr{V} = -2i(V + 1) = -2i\sqrt{1 - \mathbf{E}^2 - \mathbf{H}^2 + \mathbf{S}^2} \quad \text{..(55, A)}$$

and show that,

$$\left.\begin{aligned} \frac{\partial \mathscr{V}}{\partial \mathscr{G}} &= \mathscr{F}^* \\ \frac{\partial \mathscr{V}}{\partial \mathscr{G}^*} &= -\mathscr{F} \end{aligned}\right\} \quad \text{..} \quad \text{..} \quad \text{..} \quad \text{..} \quad \text{..} \quad (56)$$

(53) and (56) are the analogues of

$$\left.\begin{aligned} \frac{\partial \mathscr{L}}{\partial \mathscr{F}} &= \mathscr{G}^* \\ \frac{\partial \mathscr{L}}{\partial \mathscr{G}} &= \mathscr{F}^* \end{aligned}\right\} \quad \text{..} \quad \text{..} \quad (3)$$

and,

$$\left.\begin{aligned} \frac{\partial \mathscr{H}}{\partial \mathscr{F}^*} &= \mathscr{G} \\ \frac{\partial \mathscr{H}}{\partial \mathscr{G}} &= \mathscr{F} \end{aligned}\right| \; \mathscr{H} = \mathscr{L} \quad \text{..(3, A)}$$

(3) and (3, A) are Lorentz-invariant as is evident from the tensor form of the relations (6) and (11). Just as I have elsewhere[9] deduced the Lorentz-

[9] *Proc. Ind. Acad. Sci.*, A, 1936, **4**, 436.

invariance of the field equations derived from U and V in Born's theory, by using semi-vectors, it is possible to deduce that (53) and (56) are also Lorentz-invariant. It is also easily shown that these relations are invariant against Schrödinger's γ-transformations. For example, if on the right-hand sides of (53, A) and (53, B) we replace $\mathcal{F}$ and $\mathcal{F}^*$ by $e^{i\gamma}\,\mathcal{F}$ and $e^{-i\gamma}\,\mathcal{F}^*$ the expressions are multiplied respectively by $e^{-i\gamma}$ and $e^{i\gamma}$, as they should in consonance with (53). Similarly (56) also is invariant against γ-transformations.

Let us now make a transformation to the Lorentz-frame in which all the four composing three vectors are parallel, then from (55) and (55, A)

$$\left.\begin{aligned}\mathcal{U} &= -\,2\,i\,\sqrt{1 + \mathbf{B}^2 + \mathbf{D}^2}\\ \mathcal{V} &= -\,2\,i\,\sqrt{1 - \mathbf{E}^2 - \mathbf{H}^2}\end{aligned}\right\} \qquad (57)$$

and if the corresponding "mixture" field be taken as the standard one, *i.e.*, if we do not use any further γ-transformation to abolish either the electric or magnetic field quantities, we have[10]

$$\mathbf{B}^2 + \mathbf{D}^2 = (1 - \mathcal{A}^2)/\mathcal{A}^2$$

and $\mathbf{E}^2 + \mathbf{H}^2 = 1 - \mathcal{A}^2$, so that

$$\left.\begin{aligned}\mathcal{U} &= -\,2\,i/\mathcal{A}\\ \mathcal{V} &= -\,2\,i\,\mathcal{A}\end{aligned}\right\} \qquad (58)$$

or $$\frac{\mathcal{V}}{\mathcal{U}} = \mathcal{A}^2 \qquad (59)$$

If $\mathcal{U}$ and $\mathcal{V}$ be interpreted as proportional to the "displacement energy" and "field energy" respectively, this shows that they are in the ratio of $\mathcal{A}^2 : 1$, pointing out again another *dissymmetry between field and displacement.*

Treating the *singular case* of Schrödinger, the equation (52) shows that when $(\mathcal{F}\mathcal{G}) = 0$, we should have

$$\mathcal{F}^2 + \mathcal{G}^2 = 0, \text{ and } \mathcal{F}^2 = 0$$

in order that $\mathcal{U}$ may not become infinite. These equations give

$$\mathcal{G} = \pm\, i\,\mathcal{F},\ \mathcal{F}^2 = 0$$

and lead, just as in Schrödinger's article, to

$$|\mathbf{B}| = |\mathbf{D}|, \text{ and } \mathbf{B} \perp \mathbf{D}$$

when the case of infinitely weak fields is discarded.

Finally coming to *normal and abnormal fields*, these arise in the present representation from the two values of the square root as can be seen from (57). This equation shows that the sign of $\mathcal{U}$ or $\mathcal{V}$ does not depend upon the fact of either $\mathcal{A} > 0$ or $\mathcal{A} < 0$, but on taking the positive or negative sign of the square root. $\mathcal{U}$ or $\mathcal{V}$ could be positive imaginary or negative imaginary just like $\mathcal{L}$ but here for a different reason.

[10] See Reference (1); footnote on p. 472.

P.T.O.

(8)

COMPLEX REPRESENTATION - II

(Biquaternion representation).

1. Introduction.

In a previous paper[1] I have considered in detail the complex representation of Schrodinger and investigated its relationship with the ordinary representation in the earliest form of Born's field theory. Later Weiss has given another complex formalism which is valid not merely for this form of action-function but for general types of Born's Electro-dynamics. I have shown elsewhere[2] that this formalism arises quite naturally when we introduce generalised action functions in Infeld's theory without putting $G = 0$. A third type of complex-representation has been given by Watson[3] for the case of Born's original field theory. He constructs four biquaternions corresponding to the four pairings $(\vec{B}, \vec{E})$; $(\vec{D}, \vec{H})$; $(\vec{D}, \vec{B})$ and $(\vec{E}, \vec{H})$ of the field vectors, obtains their tensors, and the relations between primary and secondary field vectors in the first case. I shall indicate briefly how some of his results

(1) Proc.Ind.Acad.Sci., A.1936, 4,575 - referred to as I

(2) See the paper entitled "Generalised action functions in Born's Electro-dynamics." the 6th paper in this thesis - referred to as II

(3) Watson., Trans.Roy.Soc.Canada.(ser 3), 1936,30, p.105 - referred to as III.

follow at once from the results in I. I have also given in this paper more symmetrical representations by biquaternions such that

(i) each biquaternion is expressed purely in terms of the primary field vectors in all four cases,

(ii) the tensors of the four biquaternions should be the four action-functions $L+1, H+1, \mho+1, \nabla+1$ corresponding to the four representations of Born's field theory.

(iii) the relations between primary and secondary field vectors may be exhibited in all the four cases.

Next, I show that it is also possible to set up a biquaternion corresponding to Born's action-function in the Infeld form of a sum of Lagrangian and Hamiltonian. Finally I also indicate a biquaternionic interpretation of the parameters λ and μ introduced in the generalised action-function theory (in II). The notation will be the same as in I, II and III (unless it differs from I).

2. Watson's biquaternions.

The four biquaternions given by Watson are

$$q = 1 + iG + \vec{B} - i\vec{E} \qquad (2,1)$$

$$q_1 = (1+G^2)/Tq + \vec{H} - i\vec{D} \qquad (2,2)$$

$$q_2 = (1+\vec{D})(1+\vec{B})/(1+G^2) \tag{2,3}$$

$$q_3 = (1+i\vec{H})(1-i\vec{E}) \tag{2,4}$$

with their tensors

$$Tq = L+1 \tag{2,1a}$$

$$Tq_1 = 1-iQ \tag{2,2a}$$

$$Tq_2 = U+1 \tag{2,3a}$$

$$Tq_3 = \sqrt{1+\vec{B}^2+\vec{D}^2+G^2} \tag{2,4a}$$

(It is doubtful if the value of Tq_3 is correct. It ought to be $(U+1)\sqrt{(1+G^2)}$).

Writing $q = r + it = \gamma + i\delta$, where

$$\left.\begin{aligned} r &= 1+\vec{B}; \; t = G-\vec{E} \\ \gamma &= 1-i\vec{E}; \; \delta = G-i\vec{B} \end{aligned}\right\}$$

the relations connecting $\vec{D}, \vec{H}$, with $\vec{B}, \vec{E}$ are obtained as

$$\left.\begin{aligned} \vec{D} &= \tfrac{1}{2}\{\alpha, r\} \\ i\vec{H} &= \tfrac{1}{2}\{\beta, r\} \end{aligned}\right\} \tag{2,5}$$

where $\alpha = \dfrac{Kt}{Tq}$; $\beta = \dfrac{K\delta}{Tq}$, and $\{\ \}$ denotes the commutator and Kt the conjugate quaternion.

It might be pointed out how some of the results in Watson's paper are consequences of the equations in I. In fact equations (9), (15), (16) of the former are

identical with equations (32), (34), (40) of the latter.

From equations (2,1) - (2,4), it is easily seen that q_1, q_2 and Tq_3 which are related to the $(\vec{D}, \vec{H})$ $(\vec{D}, \vec{B})$ and $(\vec{E}, \vec{H})$ representations involve G, Tq which belongs to the first representation. lso the tensors (2,2a) and (2,4a) do not represent the action functions $H+1$, and $V+1$. Further, relations analogous to (2,5) in the other three cases cannot be derived from the above biquaternion representations.

3. Alternative biquaternions.

We shall first set up alternatives to (2,1) - (2,2). Let

$$p = q = 1 + iG + \vec{B} - i\vec{E} \qquad (3,1)$$

and

$$p_1 = 1 - iQ + \vec{D} + i\vec{H} \qquad (3,2)$$

The representation (3,1) is the same as (2,1) and (3,2) corresponds to taking $(\vec{D}, \vec{H})$ as primary vetors. Operating on p_1 with the bi-scalar $(1+iQ)/(H+1)$, we get

$$q^1 = L+1 + \vec{E} + i\vec{B}$$

if we observe that

$$Tp_1 = H+1 \qquad (3,2a)$$

and

$$(L+1)(H+1) = (1+G^2) = (1+Q^2). \qquad (I, (18))$$

We can also immediately observe the relations between q, q_1, p_1 and q'. In fact

$$\left.\begin{aligned} Tq = Sq' \;;\; Tq' = Sq \\ Tq_1 = Sp_1 \;;\; Tp_1 = Sq_1 \end{aligned}\right\} \qquad (3,4)$$

i.e. q_1, p_1 are such that the tensor of each biquaternion is the scalar of the other.

Next, to express $\vec{B}, \vec{E}$ in terms of $\vec{D}, \vec{H}$ we can write

$$\left.\begin{aligned} & p_1 = \tau_1 + i t_1 = \gamma_1 + i\delta_1 \\ \text{where} \quad & \tau_1 = 1 + \vec{D},\ t_1 = -Q + \vec{H} = -G + \vec{H} \\ & \gamma_1 = 1 + i\vec{H},\ \delta_1 = -iQ + \vec{D} = -iG + \vec{D} \end{aligned}\right\} \qquad (3,5)$$

Putting further,

$$\frac{Kt_1}{Tp_1} = \alpha_1 \;;\; \frac{K\delta_1}{Tp_1} = \beta_1$$

we can deduce, just as in Watson's paper,

$$\left.\begin{aligned} \frac{1}{2}\left\{\alpha_1, \tau_1\right\} = -\vec{B} \\ \frac{1}{2}\left\{\beta_1, \tau_1\right\} = -\vec{E} \end{aligned}\right\} \qquad (3,6)$$

Also,

$$\left.\begin{aligned} \vec{S} &= \frac{1}{2}\left[\alpha_1, \tau_1\right] = \frac{1}{2} i \left[\beta_1, \tau_1\right] \\ (\overline{U}+1)^2 &= \tau_1 K\tau_1 \left(1 + \alpha_1 K\alpha_1\right) \\ (\overline{V}+1)^2 &= \gamma_1 K\gamma_1 \left(1 + \beta_1 K\beta_1\right) \end{aligned}\right\} \qquad (3,7)$$

We go now to biquaternions alternative to (2,3)-(2,4) and set up

$$p_2 = \sqrt{1-2M} + \vec{B} + \vec{D} + \vec{S} \tag{3,8}$$

$$p_3 = \sqrt{1+2N} + i\vec{E} + i\vec{H} + \vec{S} \tag{3,9}$$

where (as in I),

$$M = (\vec{B}\,\vec{D})\,,\ N = (\vec{E}\,\vec{H})\,;\ \vec{S} = (\vec{D}\times\vec{B}) = (\vec{E}\times\vec{H})$$

It will be seen that p_2 and p_3 are expressed only in terms of the primary vectors $(\vec{B},\vec{D})$ and $(\vec{E},\vec{H})$ respectively. Also

$$T^2 p_2 = 1 + \vec{B}^2 + \vec{D}^2 + \vec{S}^2, \quad \text{since } (\vec{B}\,\vec{S}) = (\vec{D}\,\vec{S}) = 0$$

$$\left.\begin{aligned} \text{ie } Tp_2 &= U + 1 \\ \text{Similarly } Tp_3 &= V + 1 \end{aligned}\right\} \tag{3,10}$$

To express the secondary vectors in terms of the primary vectors, let us take (3,8) first and consider the quaternion

$$r_2 = \sigma + \vec{B} \qquad (\sigma = \sqrt{1-2M})$$

and the bi-vector

$$t_2 = \vec{D} + \vec{S}$$

so that

$$q_2 = r_2 + t_2$$

Then

$$r_2 K t_2 = -(\sigma + \vec{B})(\vec{D} + \vec{S}) = -\vec{E}u - (\sigma - 1)(\vec{D} + \vec{S}) + M,$$

using the relation,

$$u\vec{E} = \vec{D} + (\vec{B} \times \vec{S}) \qquad \text{(See I, p. 58, (31))}.$$

Similarly,

$$K t_2 r_2 = -(\vec{D} + \vec{S})(\sigma + \vec{B})$$

$$= u\vec{E} - (\sigma + 1)(\vec{D} + \vec{S}) + M$$

Hence

$$(\sigma+1) r_2 K t_2 - (\sigma - 1) K t_2 r_2 = -2\sigma u \vec{E} + 2M$$

$$\text{ie } \sigma(r_2 K t_2 - K t_2 r_2) + (r_2 K t_2 + K t_2 r_2) = -2\sigma u \vec{E} + 2M$$

$$\text{ie } -u\vec{E} = \frac{1}{2}\{r_2, K t_2\} + \frac{1}{2\sigma}[r_2, K t_2] - \frac{M}{\sigma} \qquad (3,11)$$

Again, let $\gamma_2 = \sigma + \vec{D}$, $\delta_2 = \vec{B} + \vec{S}$ we have

$$\gamma_2 K \delta_2 = -(\sigma + 1)(\vec{B} + \vec{S}) + u\vec{H} + M$$

using

$$u\vec{H} = \vec{B} - (\vec{D} \times \vec{S})$$

Also,

$$K\delta_2 \gamma_2 = -(\sigma - 1)(\vec{B} + \vec{S}) - u\vec{H} + M$$

Hence,

$$(\sigma - 1)\gamma_2 K \delta_2 - (\sigma + 1) K \delta_2 \gamma_2 = 2\sigma u \vec{H} - 2M$$

$$\text{ie } u\vec{H} = \frac{1}{2\sigma}\{\gamma_2, K\delta_2\} - \frac{1}{2}[\gamma_2, K\delta_2] + \frac{M}{\sigma} \qquad (3,12)$$

Relations (3,11) and (3,12) correspond to (2,5) and (3,6) but are not so elegant.

In a similar manner we can set up relations

for the biquaternion p_3 by suitably choosing $\gamma_3, t_3, \gamma_3, \delta_3$.

4. Generalised action functions.

Let us first consider Born's action-function in the form given by Infeld as the sum of a Lagrangian and a Hamiltonian. In the case where $G \neq 0$ we can write this in the form, (See II, § 9, equation (9,6)), but for a numerical term,

$$T = \sqrt{2\phi + 2\psi + 4} \qquad (4,1)$$

where ϕ and ψ are the complex invariants given by

$$\left.\begin{aligned} \phi &= F + iG \\ \psi &= P - iQ \end{aligned}\right\} \qquad (4,2)$$

with $F = \frac{1}{2}(\vec{B}^2 - \vec{E}^2)$, $P = \frac{1}{2}(\vec{D}^2 - \vec{H}^2)$; $G = (\vec{B}\vec{E}) = Q = (\vec{D}\vec{H})$

We can also introduce the complex invariant

$$\rho = R + iS \qquad (4,3)$$

where R and S are given by

$$R = \frac{1}{2}\left\{(\vec{B}\vec{H}) - (\vec{D}\vec{E})\right\}$$
$$S = \frac{1}{2}\left\{(\vec{B}\vec{D}) + (\vec{E}\vec{H})\right\}$$

The relations between the invariants (See II, (2,18) - (2,19)) can be written in the form

$$\phi\psi = -\rho^2 \qquad (4,4)$$

We proceed to show that it is possible to set up a

biquaternion whose tensor is the action-function (4,1).

Let $q = a + \vec{B} - i\vec{E} + \vec{D} + i\vec{H}$ (4,5)

the bi-vector on the right hand side of (4,5) being the sum of the bi-vectors of p and p_1 whose tensors are the Lagrangian and Hamiltonian respectively. We have

$$T^2 q = a^2 + 2\phi + 2\psi + 2(S + iR)$$

(using $G = Q$)

$$= a^2 + 2\phi + 2\psi + 2i\rho^* \tag{4,6}$$

the $(*)$ denoting the complex-conjugate. If (4,6) should give the square of (4,1) we should have

$$a^2 + 2i\rho^* = 4$$

i.e. $a = 2\sqrt{1 - \frac{1}{2} i\rho^*}$, and the biquaternion

$$q = 2\sqrt{1 - \frac{1}{2} i\rho^*} + \vec{B} - i\vec{E} + \vec{D} + i\vec{H} \tag{4,7}$$

has its tensor equal to (4,1). It is easy to see that the complex-conjugate biquaternion q^* given by

$$q^* = 2\sqrt{1 + \frac{1}{2} i\rho} + \vec{B} + i\vec{E} + \vec{D} - i\vec{H} \tag{4,8}$$

has also the same tensor as q

Let us now consider, in the general case of action functions, an interpretation of the parameters λ and μ (II, § 6).

It is well-known that any quaternion could be reduced to a quotient of two vectors, and similarly a biquaternion can be taken as a quotient of two bivectors. Consider the quotient biquaternion

$$q' = \frac{\vec{B} - i\vec{E}}{\vec{D} + i\vec{H}} \tag{4,9}$$

and let the quaternion q be defined as the logarithm of q' i.e.

$$q = \log q' \tag{4,10}$$

then, from the definition of the logarithm of a quarternion, we have

$$\left.\begin{aligned} Sq &= \log Tq' \\ \text{and,} \quad Vq &= \log Uq' \end{aligned}\right\} \tag{4,11}$$

where S and V, denote the scalar and vector, and T and U, the tensor and versor of the quaternion.

From (4,9)

$$Tq' = \frac{T(\vec{B} - i\vec{E})}{T(\vec{D} + i\vec{H})} = \frac{\phi^*}{\psi^*} = e^{\lambda - i\mu} \qquad (\text{II}, (7,1))$$

$$\therefore \quad Sq = \lambda - i\mu. \tag{4,12}$$

Thus we have expressed the complex parameter $\lambda - i\mu$ as the scalar of a biquaternion.

There does not appear to exist any simple interpretation of the vector of q.

-----o0o-----